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INVESTIGATION OF JUNCTURE STRESS FIELDS IN MULTICELLULAR SHELL STRUCTURES

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LOCKHEED MISSILES AND SPACE COMPANY
Sunnyvale, California

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For

PROPULSION AND VEHICLE ENGINEERING LABORATORY

NASA-GEORGE C. MARSHALL SPACE FLIGHT CENTER

FOREWORD

The investigation described in this report was performed by staff members of Lockheed Missiles and Space Company in cooperation with the George C. Marshall Space Flight Center of the National Aeronautics and Space Administration under Contract NAS 8-11079. Contract technical representative was H. Coldwater. The work was conducted in Analytical and Experimental Mechanics, Aerospace Sciences Laboratory, under the supervision of J. H. Klumpp. The project was under the technical direction of E. Y. W. Tsui with associates F. A. Brogan, J. M. Massard, P. Stern, and C. E. Stuhlman.

CONTENTS

NOTATIONS

A. Tensorial Notation

$x^i (i = 1,2,3)$, $x^\alpha (\alpha = 1,2)$	Space and surface coordinates (Latin indices will take the range 1, 2, 3, and Greek indices assume the range 1, 2)
$\underset{\sim}{r}$ $(\bar{\underset{\sim}{r}})$	Spatial (surface) position vector of a point
$\underset{\sim}{g}_i$ $(\underset{\sim}{a}_\alpha)$	Covariant spatial (surface) base vectors
g_{ij}, g^{ij} $(a_{\alpha\beta}, a^{\alpha\beta})$	Covariant and contravariant spatial (surface) metric tensors
g (a, b, c)	Determinant of g_{ij} $(a_{\alpha\beta}, b_{\alpha\beta}, c_{\alpha\beta})$
δ^i_j	Kronecker delta
ϵ_{ijk} $(\bar{\epsilon}_{\alpha\beta})$	Spatial (surface) permutation tensor
Γ_{ijk}, $\Gamma_{ij}{}^k$ $(\bar{\Gamma}_{\alpha\beta\gamma}, \bar{\Gamma}_{\alpha\beta}^\gamma)$	Christoffel symbols of the first and second kinds in space (middle-surface)
$(\ )\vert_i$	Covariant spatial derivative with respect to x^i and g_{ij}
$(\ )\Vert_\alpha$	Covariant surface derivative with respect to x^α and $a_{\alpha\beta}$
τ^{ij} $(\tau^{\alpha\beta})$	Contravariant spatial (surface) stress tensor
γ_{ij} $(\bar{\gamma}_{\alpha\beta})$	Spatial (surface) strain tensor
H	Mean curvature
K	Gaussian curvature
$b_{\alpha\beta}$, $c_{\alpha\beta}$	Covariant second and third fundamental tensors of middle-surface
$\rho_{\alpha\beta}$	Surface tensor of changes of curvature
$\underset{\sim}{\omega}_\alpha$	Covariant surface vector of rotation of normal at middle-surface

$\underset{\sim}{m}_\alpha$, $\underset{\sim}{n}_\alpha$	Covariant moment and stress resultant vectors
$m^{\alpha\beta}$, $n^{\alpha\beta}$	Contravariant surface tensors of moments and stress resultants
$\underset{\sim}{p}$	Load (external plus body forces) vector
q^α	Contravariant surface tensor of transverse shear forces
$E^{ijk\ell}(\overline{E}^{\alpha\beta\lambda\eta})$	Contravariant spatial (surface) tensor of elastic moduli
$\underset{\sim}{u}$	Displacement vector
$\phi_{\alpha\beta}$	Covariant surface tensor of the rotation in the middle-surface around the normal

B. Conventional Notation

a_{ij}	Flexibility influence coefficients
a_i, b_i, c_i / A , B , C , A_j	Parameters defined in text
i , j , k , m , n	Dummy subscripts
D	Flexural rigidity = $\dfrac{Eh^3}{12(1 - \nu^2)}$
E	Modulus of elasticity
F_i	Boundary forces at station i
F^f	Boundary forces of fixed edge shell due to intermediate loads
G	Shear modulus
h	Total thickness of a shell
$\bar{h}$, $\bar{k}$	Mesh spacing in α and β coordinate directions
k_{ij}	Stiffness influence coefficients

$M_{(\ )}$, $N_{(\ )}$	Moments and stress resultants
$p_{(\ )}$	Surface or body forces
$Q_{(\ )}$	Transverse shear
r_1, r_2	Principal radii of curvature
R	Radius
s	Arc length of a curve
T	Change of temperature
$u(u_1)$, $v(u_2)$, $w(u_3)$	Displacement components in directions α , β , and z
V	Strain energy per unit area of the undeformed middle-surface
$\overline{W}$	Strain energy per unit volume of the undeformed shell
$z = x_3$	Coordinate measured normal to the middle surface of a shell
i , j , k	Unit base vectors in Cartesian coordinates
x , y	Cartesian coordinates in a plane
x , θ	Cylindrical coordinates
ϕ , θ	Spherical coordinates
α , β	Orthogonal curvilinear coordinates on middle-surface with coordinate curves along lines of principal curvatures
δ_i	Boundary displacements at a station i
$A_1 d\alpha$, $A_2 d\beta$	Line-element along α and β -curves, where A_1 , A_2 are the Lame parameters
$\epsilon_{(\ )}$, $\gamma_{(\ )}$	Direct and shear strains
χ_1 , χ_2 , χ_{12}	Changes of curvature and torsion of middle-surface

ν	Poisson's ratio
ω_1 , ω_2	Rotations of the normal at the middle-surface in the directions α and β
$(\ )_{,x^i}$	Partial derivative $\left(= \frac{\partial(\ )}{\partial x^i}\right)$, $(\ )_{,\alpha} \equiv (\ )_{,1}$, $(\ )_{,\beta} \equiv (\ )_{,2}$
$(\ )_i^j$	The function at a discrete point i , j where i , j implies the α and β directions respectively
ϵ^T	Thermal strain = coefficient of linear expansion multiplied by the change of temperature (T)
Φ	Rotation in the middle-surface around the normal

Other notations are defined in the text as required.

SUMMARY

The study described in this report is concerned with the discontinuity stress fields that arise in thin elastic multicellular shell structures subjected to inertial, pressure, and thermal loading conditions in combination with residual stresses resulting from fabrication and assembly. Since analytic and variational methods were not believed to be suitable for solving such problems, numerical techniques were investigated.

The method of analysis decided upon for the solution of the complete shell structure is similar to the "slope-deflection" procedure used in the analysis of indeterminate space structures, in that the structure is first analyzed in terms of the behavior of the simple elements, cone, sphere, cylinder, and plate, as represented by stiffness matrices which relate the boundary forces on the element to boundary displacements. From this information and necessary conditions of equilibrium and displacement compatibility between elements, a set of equations can be formed and solved to yield solutions for the actual element displacement boundary values corresponding to the continuous structure. Once the displacement boundary values for the elements are known, the stresses throughout the various elements can be determined from the stiffness functions.

The major difficulty in this method is the determination of the stiffness matrices for the individual shell elements. In the procedure described, this is accomplished through the finite difference reduction of the shell equations. The success of

the method is dependent on the ability to solve large sets of algebraic equations accurately and in reasonable computer time. A direct method for accomplishing such solutions was developed and is described.

This report includes a review of pertinent literature; the derivation of the general shell equations and their specialization to the cone, sphere, cylinder, and plate; a brief discussion of finite difference expressions; a description of the method of solution, and conclusions and recommendations.

Information presented in appendices includes, a discussion of the geometry of a specific multicellular shell structure, a discussion of the equations governing anisotropic plates and cylinders, a general discussion of residual stresses in welded structures, and a detailed description of the direct method developed for solving large matrices of finite-difference equations.

1. INTRODUCTION

1.1 Background

The rational design of large launch and space vehicles containing multicellular propellant containers requires a capability for determining the discontinuity stress fields that arise in structures composed of a combination of shell elements subjected to inertial, pressure, and thermal loading conditions together with residual stresses resulting from fabrication and assembly. Shell structures composed of dissimilar elements with some non-geodesic junctures generally cannot be analyzed with closed-form techniques. Variational or energy methods also appear to be impractical for solving such problems. Therefore, numerical solution seems to be the only practicable recourse.

The purpose of this report is to present the equations applicable to the analysis of multicellular shell structures, to describe a practicable numerical procedure for obtaining the solution of such equations by digital computer, and to discuss some practical aspects of the analysis of multicellular structures including the effects of residual stresses resulting from fabrication techniques.

1.2 Scope of the Investigation

The scope of the investigation can be summarized in terms of the contents of the individual chapters which form this report.

The first phase of the investigation includes a survey of pertinent literature. The results of the survey are presented in Chapter 2 as (1) studies in general shell theory, (2) analysis of specific shell types, and (3) analysis of multiply connected shell structures. This survey revealed no available procedure that could be applied directly to solve the type of problem under consideration.

A detailed presentation of the infinitesimal theory of thin elastic shells is contained in the third chapter. This information includes a critical review of the theory. The objective of this review was to obtain the simplest possible system of equations for development of the analysis. It is observed that, because of the special geometry and loading of the structure, the major differences of various versions of Love's first approximation disappear.

Shell equations, based on Love's first approximation with additional assumptions, are derived using tensor analysis. These equations are then translated into the conventional unabridged form, using an orthogonal coordinate system with coordinate curves along lines of principal curvature. This set of equations furnishes the basis of the method of analysis described in this report.

In the transformation of the equilibrium equations, the six coupled equations are condensed into three equations by the method of elimination. These three equations are then written in terms of displacement components, allowing for variable thickness and modulus of elasticity, and become the lengthy equations presented in the third chapter.

In this chapter, the shell equations expressed in terms of displacement components are presented not only in general form but also with coefficients specialized for spheres, cones, cylinders, and plates.

A theoretical treatment of the boundary conditions necessary for unique solution of the governing differential equations of linear shell theory is also presented in this chapter. It is shown that four boundary conditions instead of five are required when the Kirchhoff-Love approximations are used. The special cases of fixed and free edge conditions are given.

In Chapter 4 membrane theory is discussed. This is an approximation of the general theory of thin elastic shells in which all moments are assumed to be negligible. The equations for membrane shells are presented and a method of solution is outlined. It is believed that in certain situations the solution of a set of three differential equations in terms of the displacement components u , v , and w may be more desirable. These equations can be obtained directly from the general governing equations by setting the higher order terms involving the thickness equal to zero. This implies a finite extensional stiffness but negligible bending stiffness. With this approximation the governing equations for membrane shells are obtained and specialized for tapered cones and for uniform thickness cones, spheres, and cylinders.

An essential step in the solution of the multicellular shell problem described in this report is the finite difference reduction of the shell equations to algebraic form. A brief discussion of finite difference expressions is presented in Chapter 5 including the derivation from a Taylor series expansion of a function of two variables or from the equivalent polynomial expression. Central difference expressions for all derivatives occurring in the shell equations are given explicitly in terms of a "rectangular" array of mesh points. The procedure for generating similar expressions in terms of mesh points having arbitrary spacing is outlined in matrix form.

Chapter 6 contains a discussion of the general method recommended for the analysis of multicellular shell structures. This chapter includes discussions of (a) general considerations, (b) stress and deformation of shell segments under intermediate (non-edge) loads, (c) stress and deformation of shell segments due to edge loads, (d) equilibrium conditions and compatibility requirements at the juncture of shell segments, and (e) the general method of analysis.

Conclusions of the investigation and recommendations for further work are presented in Chapter 7.

Four appendices are included in this report. The first defines in detail the geometry of the specific bulkhead structure described in Procurement Request TP3-85481. This includes the necessary dimensions, coordinate systems, and intersections of the component shells which comprise the bulkhead. General expressions for dimensions are presented. The equations for a sphere, cone, cylinder, and plate are given for a system of rectangular coordinates. Then a system of orthogonal curvilinear coordinates is presented for each of the shell elements. From this information the first fundamental form and base vectors of the surfaces are given. By use of the second fundamental form the principal radii of curvature are obtained. The intersection curves of the various shell components are computed.

The derivation of equations governing the behavior of anisotropic plates and cylinders is presented in Appendix II. This information is applicable to stiffened plates and cylinders in the sense that if the stiffeners are closely spaced the structure can be approximated as an anisotropic plate or cylinder. Constitutive equations have been derived for plates and cylinders when the stiffeners are orthogonal and of equal size and spacing. Two special cases are considered: (1) stiffeners are oriented in the coordinate directions, and (2) the stiffeners are oriented 45 degrees from the coordinate directions. The corresponding constitutive equations and the governing differential equations expressed in terms of the displacement components u , v , and w are presented for these cases.

The general aspects of residual stresses in welded structures are presented in Appendix III. This includes (a) a general statement of the problem, (b) methods of evaluating residual stresses, (c) residual stress or strain patterns, and (d) discussion and recommendations for further work. As a result of the study of residual stresses, it is conceivable that the induced residual stresses in the non-stress relieved welded vessels under consideration may be significant. Further, it is almost impossible to predict residual stress magnitudes and distributions analytically. Consequently, it is recommended that certain experiments regarding residual stress magnitude and distribution should be performed so that an accurate determination of welded joint efficiency and attenuation length can be made and incorporated into the theoretical analysis.

The success of the method described in this report for the numerical solution of the shell equations depends on a procedure for solving large sets of finite difference equations accurately and in reasonable computer time. A direct method for accomplishing such solutions is described in Appendix IV.

2. LITERATURE SURVEY

A literature search has been undertaken to assess the methods of analysis of multiply connected plate and shell structures. In particular, concern is focused around the juncture of conical, spherical, cylindrical shell segments and flat plates.

Because the structure under consideration is unusual, the literature was evaluated by work according to the following classifications:

1. General Shell Theory
2. Specific Shell Geometries
3. Multiply Connected Shell Structures

The bibliography in Chapter 8 is also divided into these three subject headings.

2.1 General Shell Theory

No attempt will be made to trace the evolution of shell theory up to the present since this can be found in (A.8)*, (A.14), (A.15) and certain references in the bibliography of shell and shell-like structures by W. A. Nash (A.12). Furthermore, attention will be directed to the theory of small deflections of thin elastic isotropic shells. By small deformations we assume that the equilibrium conditions for deformed elements are the same as if they were not deformed (A.14). The equilibrium equations in terms of stress resultants in this theory present no difficulty and their general expressions have been derived by various investigators (A.1), (A.2), (A.3), (A.4), (A.6), (A.12), (A.13), etc. Similarly, expressions for strain-displacement relations present no difficulty.

*Numbers within parentheses refer to literature listed in the Bibliography.

The essential problem in the theory has been in the formulation of appropriate constitutive equations or stress-strain relations. For sufficiently thin isotropic shells, a simple system of stress-strain equations can be formulated. Such a system is known as "Love's First Approximation." Based on the Kirchhoff-Love assumptions (A.4), (A.6), different versions of Love's first approximation have been derived (A.1), (A.3), (A.4), (A.6), (A.10), (A.13), (A.14), (A.15), (A.16), etc. It can be easily observed, unfortunately, that considerable differences occur with various workers, particularly as characterized by their expressions for the changes of curvatures. Some of these expressions also violate certain invariance requirements, such as the equilibrium condition for moments about the normal direction of a shell element, for example. It should be pointed out, however, that for practical applications discrepancies in different forms of the constitutive equations mentioned above are immaterial as long as the rotation in the middle-surface about the normal is small (A.4).

2.2 Specific Shell Geometries

A review now will be made of the application of shell theory to the stress analysis of specific shell geometries. In the formulation of the governing differential equations for shells, a choice is possible as to the dependent variables to be employed. These variables may be expressed in terms of:

A. Displacements,

B. Stress Resultants, or

C. A combination of Displacements and Stress Resultants.

Examples of this development are found in (A.1), (A.3), (A.13), (A.16). Since no general analytic method is available to solve the governing differential equations, various methods of approach have been used in the solution of shells having specific geometries, loadings and boundary conditions. These methods include:

a. Exact

b. Asymptotic

c. Variational

d. Numerical

Due to the fact that the structure to be investigated is composed of classical shell elements, namely, plates, cones, spheres and cylinders, attention will be restricted to these geometries which are essentially segments of shells of revolution.

An exact solution in hypergeometric series can be obtained for spherical shell segments axisymmetrically loaded at the edges (A.1), (A.13). Values of these series for certain range of radius-to-thickness ratio have been recently tabulated (B.15). Asymptotic solutions can also be obtained for this case and for shells with an arbitrary meridional curve (B.12). For shells of revolution loaded asymmetrically and having two boundaries which can be described by a function of one of the coordinates, solutions can be obtained by the use of Fourier series to reduce the partial differential equations of equilibrium to ordinary differential equations that can then be solved by asymptotic methods. Examples of this approach are found in (A.1), (B.2), (B.5), (B.10).

Variational methods, specifically, the method of Ritz or Galerkin, have not been applied too widely to two-dimensional problems. This is probably because the trial functions of the dependent variable will be in the form of a double series which usually converge very slowly. This method has been applied to the intersection of two cylindrical shells (C.6).

Numerical solutions have become practical because of the availability of high-speed digital computers. One of the numerical methods commonly used involves writing the governing differential equations in difference form to yield a system of linear algebraic equations. This system of equations is then solved by standard methods for the dependent variables. On the other hand, numerical integration can also be used in the solution of the governing equations (B.20). The Finite-Difference Method has been applied by many investigators to shells having axisymmetric loads, one such example is given in (B.18). The case of asymmetric loads can also be handled by the Fourier series method noted previously combined with the numerical analysis. For shells having arbitrary boundaries and loads, numerical analysis seems to be the only practical recourse for these problems at the present. It is interesting to note that available literature on the numerical solutions for arbitrary shell segments is mostly confined to the plate problems (A.1), (B.4).

2.3 Multiply Connected Shell Structures

The development of stress analysis methods for specific plate and shell elements subjected to given surface loads and edge loads, as described in section 2.2, is of importance when these elements are employed as the main structure or when they are combined into one integral structure. In the latter case, one will find shell elements having different geometrical properties. Because of these geometrical discontinuities the structure is analyzed by cutting along these discontinuities and then applying edge loads to the cut elements in order that displacements at the shell interfaces be compatible.

Perhaps the simplest method for "joining" shell elements together is found when studying shells of revolution subjected to axisymmetrical loads. In this case use is made of influence coefficients to insure compatibility of displacements and stresses at the shell juncture. This method is described in (A.1), (A.14), (A.17), (C.3), (C.4), (C.5). This method of analysis has the advantage that is quite systematic once the influence coefficients have been determined.

For shells of revolution subjected to asymmetric loading the problem of juncture stresses becomes more difficult. The technique is to develop a solution with arbitrary edge loads which can be expressed in terms of Fourier series. Then the juncture stresses between shells are obtained through the compatibility condition between loads and displacements which determines the coefficients in the Fourier series. This method is used in plate bending analysis (A.17) and is described for shell problems in (A.1), (A.2), (A.14). In determining the effect of edge loads the general problem is reduced to solving ordinary differential equations after separation of variables. This separation of variables is possible, however, only when the boundary edges are a function of one coordinate.

The solution for cylindrical shells has been extensively explored for use in roof structures (C.2). Analytic solution is obtained in terms of the radial displacement w of an 8th order partial differential equation. This solution satisfies conditions for arbitrary boundaries along the generators and for simple supports on the other edges.

3. INFINITESIMAL THEORY OF THIN ELASTIC SHELLS

3.1 Introduction

In order to evaluate the stresses and deformations of the multicellular shell structures with reasonable accuracy, it is essential that a set of basic shell equations be established. These equations include the stress-strain relations, strain-displacement relations, compatibility and equilibrium equations and the constitutive equations which relate the moments and stress resultants in terms of changes of curvature and strains.

A simple and yet consistent set of shell equations is obtainable through the theory known as Love's first approximation. This theory is based on the following well-known Kirchhoff-Love hypothesis:

a. Normal stress (σ_3) and shear strains $(\gamma_{13}, \gamma_{23})$ can be neglected.

b. The shell is thin, i.e., $\frac{h}{r} \ll 1$ where h is the thickness of shell and r is the minimum principal radius of curvature.

c. Normals to the undeformed middle-surface remain normals to the deformed middle-surface.

To achieve a simplified formulation, facilitating the expected numerical analysis and conforming to the invariance requirements, additional assumptions are made:

d. Deformations are small.

e. Materials are homogeneous, isotropic and behave elastically within the stress field.

f. Rotation in the middle-surface around the normal is small.

It is noted that the last assumption is justified in view of the symmetry in geometry and loading of the structure under consideration.

In what follows the fundamentals of the differential geometry of a surface which are essential to the classical shell elements are described first. Then, derivation of the general shell equations is outlined and the relevant expressions are given in a compact form using tensor notation. Finally, the general equations of the linear theory of thin elastic shells under the above mentioned assumptions are expressed in terms of the physical parameters. This set of equations, consequently, provides the basic information for development of the numerical analysis.

3.2 Fundamentals of Differential Geometry of a Surface

A surface in a three-dimensional Euclidean space is defined as the locus of a point whose position vector $\bar{\underset{\sim}{r}}$, relative to some reference origin 0, is a function of two arbitrary curvilinear coordinates x^{α} $(\alpha = 1, 2)$. In terms of right-handed orthogonal Cartesian coordinates x as shown in Fig. 3-1, one has

$$x^i = x^i(x^{\alpha}) \tag{3-2-1}$$

and

$$\underset{\sim}{r} = \bar{\underset{\sim}{r}} + z\underset{\sim}{a}_3 \tag{3-2-2}$$

where $\underset{\sim}{r}$ is a spatial position vector and $\underset{\sim}{a}_3$ is the unit normal vector.

The square of an arc element is given by the scalar product of $d\bar{\underset{\sim}{r}}$, namely

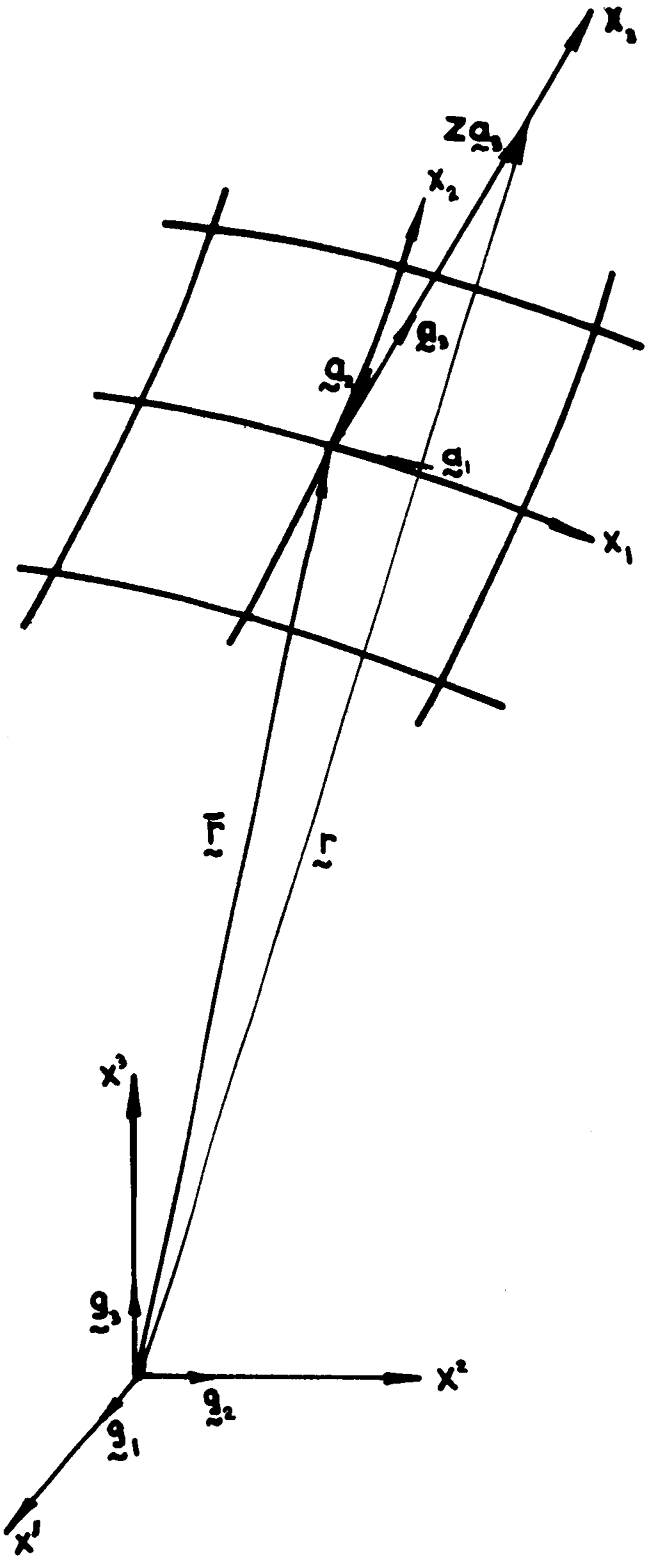


FIG. 3-1 COORDINATE SYSTEMS

$$ds^2 = d\underset{\sim}{r} \cdot d\underset{\sim}{r} = a_{\alpha\beta} dx^\alpha dx^\beta \quad , \tag{3-2-3}$$

where

$$a_{\alpha\beta} = a_{\beta\alpha} = \underset{\sim}{a}_\alpha \cdot \underset{\sim}{a}_\beta \quad , \qquad \underset{\sim}{a}_\alpha = \underset{\sim}{r}_{,\alpha} \; , \tag{3-2-4}$$

and $a_{\alpha\beta}$ is the metric tensor and $\underset{\sim}{a}_\alpha$, $\underset{\sim}{a}_\beta$ are the covariant base vectors respectively. Equation (3-2-3) is called the <u>first fundamental form</u> of the surface. The corresponding conjugate tensor and contravariant base vectors are given by

$$a^{\alpha\beta} = a^{\beta\alpha} = \underset{\sim}{a}^\alpha \cdot \underset{\sim}{a}^\beta \quad , \qquad \underset{\sim}{a}^\alpha = a^{\alpha\beta} \underset{\sim}{a}_\alpha \, . \tag{3-2-5}$$

It is noted that equations (3-2-4) and (3-2-5) also satisfy the following relations

$$\underset{\sim}{a}^\alpha \cdot \underset{\sim}{a}_\beta = a^{\alpha\eta} a_{\eta\beta} = \delta^\alpha_\beta = \begin{cases} 1 \; , & \alpha = \beta \; , \\ 0 \; , & \alpha \neq \beta \; , \end{cases} \tag{3-2-6}$$

where δ is the Kronecker delta.

The vector product of the covariant base vectors is expressed as

$$\underset{\sim}{a}_\alpha \times \underset{\sim}{a}_\beta = \bar{\epsilon}_{\alpha\beta} \underset{\sim}{a}_3 \quad , \tag{3-2-7}$$

where

$$\bar{\epsilon}_{\alpha\beta} = (\underset{\sim}{a}_\alpha \times \underset{\sim}{a}_\beta) \cdot \underset{\sim}{a}_3 = a^{\frac{1}{2}} e_{\alpha\beta} \tag{3-2-8}$$

and

$$a = | a_{\alpha\beta} | \; , \; e_{11} = e_{22} = 0 \; , \; e_{12} = -e_{21} = 1 \tag{3-2-9}$$

Similar expressions for the contravariant base vectors may be obtained simply by raising the indices of equation (3-2-7).

The <u>second and third fundamental forms</u> of the surface are respectively defined by the following scalar products

$$d\underset{\sim}{r} \cdot d\underset{\sim}{a}_3 = -b_{\alpha\beta} dx^\alpha dx^\beta \quad , \tag{3-2-10}$$

$$d\underset{\sim}{a}_3 \cdot d\underset{\sim}{a}_3 = b_{\alpha\eta} b^\eta_\beta dx^\alpha dx^\beta \quad , \tag{3-2-11}$$

where

$$b_{\alpha\beta} = b_{\beta\alpha} = \underset{\sim}{a}_3 \cdot \underset{\sim}{a}_{\alpha,\beta} \quad , \qquad b_{\alpha\eta} b^\eta_\beta = \underset{\sim}{a}_{3,\alpha} \cdot \underset{\sim}{a}_{3,\beta} \tag{3-2-12}$$

The mean curvature (H) and Gaussian curvature (K) of the surface are invariants which are expressed in terms of the covariant second fundamental surface tensors, as follows

$$H = \frac{1}{2} b^\alpha_\alpha = -\frac{1}{2}\left(\frac{1}{r_1} + \frac{1}{r_2}\right) \quad , \tag{3-2-13}$$

$$K = \frac{|b_{\alpha\beta}|}{a} = \frac{1}{2} \delta^{\alpha\beta}_{\lambda\eta} b^\lambda_\alpha \, b^\eta_\beta = b^1_1 b^2_2 - b^1_2 b^2_1 \; .$$

$$= \frac{1}{r_1 r_2} \quad , \tag{3-2-14}$$

where r_1 , r_2 are the principal radii of curvature and

$$\delta^{\alpha\beta}_{\lambda\eta} = \bar{\epsilon}^{\alpha\beta} \bar{\epsilon}_{\lambda\eta} \quad , \quad \delta^{\alpha\eta}_{\beta\eta} = \delta^\alpha_\beta \qquad \delta^{\lambda\eta}_{\lambda\eta} = 2 \tag{3-2-15}$$

If $\underset{\sim}{g}_i$ are the spatial covariant base vectors, then their first derivatives may be written as

$$\underset{\sim}{g}_{i,j} = \Gamma_{ijn} \underset{\sim}{g}^n = \Gamma^k_{ij} \underset{\sim}{g}_k \tag{3-2-16}$$

where the Christoffel symbols of the first kind (Γ_{ijn}) and second kind (Γ^k_{ij}) in space are defined, in terms of the covariant differentiations of the spatial metric tensors, respectively by

$$\Gamma_{ijn} = \frac{1}{2}\left(g_{in,j} + g_{jn,i} - g_{ij,n}\right) \quad , \tag{3-2-17}$$

$$\Gamma_{ij}^{\kappa} = g^{ij}\Gamma_{ij\kappa} \; . \tag{3-2-18}$$

It can be shown (A.2) that partial differentiations of a vector $\underset{\sim}{u}$ in space are given by

$$\underset{\sim}{u}_{,j} = u^{\lambda}\big|_j \, \underset{\sim}{g}_{\lambda} = u_{\lambda}\big|_j \, \underset{\sim}{g}^{\lambda} \quad , \tag{3-2-19}$$

where

$$u^{\lambda}\big|_j = u^{\lambda}_{,j} + \Gamma_{\kappa j}^{\lambda} u^{\kappa} \quad , \tag{3-2-20}$$

and

$$u_{\lambda}\big|_j = u_{\lambda,j} - \Gamma_{\lambda j}^{\kappa} u_{\kappa} \; . \tag{3-2-21}$$

Equations (3-2-20) and (3-2-21) represent respectively the <u>covariant derivatives</u> of the components u^{λ} , u_{λ} of the vector $\underset{\sim}{u}$.

For a given surface, the tensors $a_{\alpha\beta}$, $b_{\alpha\beta}$ may be evaluated by equations (3-2-3) and (3-2-10). It can be verified that they are related by the equations of Codazzi and Gauss which can be written respectively

$$b_{\alpha\beta}\|_{\eta} - b_{\alpha\eta}\|_{\beta} = 0 \quad , \tag{3-2-22}$$

$$b_{\alpha\eta}b_{\beta\lambda} - b_{\alpha\beta}b_{\eta\lambda} = \bar{R}_{\lambda\alpha\beta\eta} \quad , \tag{3-2-23}$$

where $\bar{R}_{\lambda\alpha\beta\eta}$ stands for the Riemann-Christoffel surface tensor which can be written

$$\bar{R}_{\lambda\alpha\beta\eta} = a_{\lambda\gamma}\bar{R}^{\gamma}_{\alpha\beta\eta}$$

$$= \bar{\Gamma}^{\gamma}_{\alpha\eta,\beta} - \bar{\Gamma}^{\gamma}_{\alpha\beta,\eta} + \bar{\Gamma}^{\mu}_{\alpha\eta}\bar{\Gamma}^{\gamma}_{\mu\beta} - \bar{\Gamma}^{\mu}_{\alpha\beta}\bar{\Gamma}^{\gamma}_{\mu\eta} , \qquad (3\text{-}2\text{-}24)$$

and the covariant derivatives of the surface tensors as shown in equation (3-2-22) read

$$b_{\alpha\beta}\|_{\gamma} = b_{\alpha\beta,\gamma} - \bar{\Gamma}^{\lambda}_{\alpha\gamma} b_{\lambda\beta} - \bar{\Gamma}^{\lambda}_{\beta\gamma} b_{\alpha\lambda} \quad . \qquad (3\text{-}2\text{-}25)$$

3.3 Derivation of Shell Equations

A. Stress-Strain Relations

For a linearly elastic, homogeneous and isotropic body, the spatial (contravariant) stress tensor can be related to the (covariant) strain tensor by the following condition

$$\tau^{ij} = \frac{1}{2}\left(\frac{g}{\mathfrak{g}}\right)^{\frac{1}{2}}\left(W,\gamma_{ij} + W,\gamma_{ji}\right) \quad , \qquad (3\text{-}3\text{-}1)$$

where g , $\mathfrak{g}$ are respectively the determinants of the spatial metric tensors of the undeformed and deformed body, and W is the strain energy per unit volume of the undeformed shell. If the deformations are small, $g \cong \mathfrak{g}$ and the following expression may be assumed for the invariant

$$W = \frac{1}{2} E^{ijkl}\gamma_{ij}\gamma_{kl} \quad , \qquad (3\text{-}3\text{-}2)$$

and equation (3-3-1) reads $\tau^{ij} = E^{ijkl}\gamma_{kl}$, (3-3-3)

where

$$E^{ijkl} = G\left(g^{ik}g^{jl} + g^{il}g^{jk} + \frac{2\nu}{1-2\nu} g^{ij}g^{kl}\right) , \qquad (3\text{-}3\text{-}4)$$

and

$$G = \frac{E}{2(1+\nu)} \qquad (3\text{-}3\text{-}5)$$

It is noted that the tensor equation (3-3-3) is based on the general spatial coordinates. It can be expressed in surface tensors through a suitable transformation. If assumption (a) of section 3-1 is observed, it follows that

$$\tau^{\alpha\beta}=E^{\alpha\beta\rho\lambda}\gamma_{\rho\lambda} \quad , \tag{3-3-6}$$

where

$$E^{\alpha\beta\rho\lambda}=G\left(a^{\alpha\lambda}a^{\beta\rho}+a^{\alpha\rho}a^{\beta\lambda}+\frac{2\nu}{1-2\nu}a^{\alpha\beta}a^{\rho\lambda}\right) \quad . \tag{3-3-7}$$

B. Strain-Displacement Relations

The strain tensor is a function of the metric tensors of the deformed and undeformed body, namely

$$\gamma_{ij}=\frac{1}{2}\left(\mathcal{G}_{ij}-g_{ij}\right) \quad . \tag{3-3-8}$$

Observing that

$$\mathcal{G}_{ij}=\underset{\sim}{G}_i\cdot\underset{\sim}{G}_j=g_{ij}+\underset{\sim}{g}_i\cdot u_{,j}+\underset{\sim}{g}_j\cdot\underset{\sim}{u}_{,i}+\underset{\sim}{u}_{,i}\cdot\underset{\sim}{u}_{,j} \quad , \tag{3-3-9}$$

where $\underset{\sim}{u}$ is the displacement vector, equation (3-3-8) becomes

$$\begin{aligned}\gamma_{ij}&=\frac{1}{2}\left(\underset{\sim}{g}_i\cdot u_{,j}+\underset{\sim}{g}_j\cdot u_{,i}+u_{,i}\cdot u_{j}\right)\\ &\cong\frac{1}{2}\left(\underset{\sim}{g}_i\cdot u_{,j}+\underset{\sim}{g}_j\cdot u_{,i}\right) \quad .\end{aligned} \tag{3-3-10}$$

Using equations (3-2-19) through (3-2-21) inclusive, and noticing $\underset{\sim}{u}=u_\eta \underset{\sim}{g}^\eta$ the desired strain-displacement relations are obtained (in general coordinates) as follows

$$\gamma=\frac{1}{2}\left(u_i|_j+u_j|_i\right) \quad . \tag{3-3-11}$$

The corresponding equation in covariant surface derivatives is

$$\bar{\gamma}_{\alpha\beta} = \frac{1}{2}\left(u_{\alpha\|\beta} + u_{\beta\|\alpha}\right) - u_3 b_{\alpha\beta} \quad ; \tag{3-3-12}$$

and the covariant surface tensor of the rotation of the normal at the middle-surface

$$\omega_\alpha = u_3\|_\alpha + u_\lambda b^\lambda_\alpha \quad . \tag{3-3-13}$$

The change of curvature tensor is obtained from the covariant differentiation of equation (3-3-12), i.e.,

$$\rho_{\alpha\beta} = -\bar{\gamma}_{\alpha\beta|3} = \omega_{\alpha\|\beta} + \omega_{\beta\|\alpha} + b^\eta_\beta \phi_{\alpha\eta} + b^\eta_\alpha \phi_{\beta\eta} \quad , \tag{3-3-14}$$

where

$$\phi_{\alpha\beta} = \frac{1}{2}\left(u_{\beta\|\alpha} - u_{\alpha\|\beta}\right) . \tag{3-3-15}$$

The above antisymmetric surface tensor $\phi_{\alpha\beta}$ describes the rotation in the middle-surface around the normal.

C. Equilibrium Equations

Let $\underset{\sim}{p}$ represent the load vector, measured per unit area of the middle-surface, and $\underset{\sim}{n}_\alpha$, $\underset{\sim}{m}_\alpha$ are the respective stress resultants and moments, measured per unit length of the line x_α = constant , $x_3 = 0$ along the boundary of an element. These vectors may be denoted by

$$\underset{\sim}{n}_\alpha = (a^{\alpha\alpha})^{-\frac{1}{2}}(n^{\alpha\lambda}\underset{\sim}{a}_\lambda + q^\alpha \underset{\sim}{a}_3) \quad , \tag{3-3-16}$$

$$\underset{\sim}{m}_\alpha = (a^{\alpha\alpha})^{-\frac{1}{2}}(m^{\alpha\lambda}\underset{\sim}{a}_3 \times \underset{\sim}{a}_\lambda) \quad , \tag{3-3-17}$$

where

$$n^{\alpha\beta} = \int_{-\frac{h}{2}}^{\frac{h}{2}} (1 - 2zH + z^2K)\sigma^{\alpha\beta} dz \ , \tag{3-3-18}$$

$$m^{\alpha\beta} = \int_{-\frac{h}{2}}^{\frac{h}{2}} (1 - 2zH + z^2K)\sigma^{\alpha\beta} z \, dz \ , \tag{3-3-19}$$

$$q^{\alpha} = \int_{-\frac{h}{2}}^{\frac{h}{2}} (1 - 2zH + z^2K)\tau^{\alpha 3} dz \ , \tag{3-3-20}$$

the quantities H and K are defined by equations (2-3-14) and (2-3-15) respectively.

The condition of static equilibrium of an element requires that (1) the vector sum of all forces and (2) the moment of all forces about an arbitrary point vanish. Consequently,

$$(\underset{\sim}{n}_{\alpha}\sqrt{a}\, a^{\alpha\alpha})_{,\alpha} + \sqrt{a}\, \underset{\sim}{p} = 0 \ , \tag{3-3-21}$$

$$(\underset{\sim}{m}_{\alpha}\sqrt{a}\, a^{\alpha\alpha})_{,\alpha} + \underset{\sim}{a}_{\alpha} \times (\underset{\sim}{n}_{\alpha}\sqrt{a}\, a^{\alpha\alpha}) = 0 \ . \tag{3-3-22}$$

If equation (3-3-16) is substituted into equation (3-3-21) and if the following well known formulas of Weigarten and Gauss

$$\underset{\sim}{a}_{\alpha,\beta} = \Gamma^{\eta}_{\alpha\beta}\underset{\sim}{a}_{\eta} + b_{\alpha\beta}\underset{\sim}{a}_{3} \ , \tag{3-3-23}$$

$$\underset{\sim}{a}^{\alpha}_{,\beta} = -\Gamma^{\alpha}_{\beta\eta}\underset{\sim}{a}^{\eta} + b^{\alpha}_{\beta}\underset{\sim}{a}_{3} \ , \tag{3-3-24}$$

$$\underset{\sim}{a}_{3,\alpha} = -b^{\eta}_{\alpha}\underset{\sim}{a}_{\eta} \ , \tag{3-3-25}$$

are employed, three equations are found by setting the coefficients of the covariant base vectors equal to zero. Then if the surface covariant derivatives are introduced, the following equilibrium equations are obtained

$$n^{\alpha\beta}\|_{\alpha} - b^{\beta}_{\alpha} q^{\alpha} + p^{\beta} = 0 \quad , \tag{3-3-26a,b}$$

$$n^{\alpha\beta} b_{\alpha\beta} + q^{\alpha}\|_{\alpha} + p^{3} = 0 \quad . \tag{3-3-26c}$$

Similarly, equation (3-3-22) can be transformed into

$$m^{\alpha\beta}\|_{\alpha} - q^{\beta} = 0 \quad , \tag{3-3-26d,e}$$

$$\bar{\epsilon}_{\lambda\beta} n^{\lambda\beta} - \bar{\epsilon}_{\lambda\beta} m^{\alpha\beta} b^{\lambda}_{\alpha} = 0 \quad . \tag{3-3-26f}$$

D. Constitutive Equations

The general expression of the strain energy per unit area of the undeformed middle-surface can be written (Ref. A.4) as

$$V = \int_{-\frac{h}{2}}^{\frac{h}{2}} (1 - 2zH + z^2 K) \overline{W}\, dz \quad , \tag{3-3-27}$$

in which $\overline{W}$ is defined by equation (3-3-2). $\overline{W}$ can be expanded into a truncated Taylor series with respect to the z- coordinate, and the spatial elastic moduli, change of curvature, and strain tensors can be expressed in terms of their surface equivalents. Integrating (3-3-27) so obtained, through the thickness, one arrives at

$$\begin{aligned} V &= \overline{V}_1 + \overline{V}_2 \\ &= \frac{1}{2} h\, \overline{E}^{\alpha\beta\lambda\eta} \left(\bar{\gamma}_{\alpha\beta} \bar{\gamma}_{\lambda\eta} + \frac{h^2}{12} \rho_{\alpha\beta} \rho_{\lambda\eta} \right) \quad . \end{aligned} \tag{3-3-28}$$

Since V is invariant under all transformations of coordinates, one can relate the stress resultants $(n^{\alpha\beta})$ and moments $(m^{\alpha\beta})$ to equation (3-3-28) in a way similar to that for the stresses (τ^{ij}) in terms of W (see subsection A above) to obtain:

$$n^{\alpha\beta} = V_{1,\bar{\gamma}_{\alpha\beta}} \quad , \tag{3-3-29a}$$

$$m^{\alpha\beta} = V_{2,\rho_{\alpha\beta}} \quad . \tag{3-3-29b}$$

Consequently, we have

$$n^{\alpha\beta} = h\, E^{\alpha\beta\lambda\eta}\, \bar{\gamma}_{\lambda\eta} \quad , \tag{3-3-30a}$$

$$m^{\alpha\beta} = \frac{h^3}{12} E^{\alpha\beta\lambda\eta}\, \rho_{\lambda\eta} \quad . \tag{3-3-30b}$$

Since the equations of Codazzi and Gauss may be used to obtain the compatibility equations for strain and change of curvature tensors, the derivation of the latter equations will not be considered.

3.4 Physical Interpretation of Shell Equations

The formulas derived in the previous section for general middle-surface coordinate systems are quite complicated when expanded. In order to perform an engineering analysis, these equations must be translated into conventional notation. For the problem under consideration, a simpler set of formulas can be developed if we adopt the orthogonal curvilinear coordinate system formed by the lines of curvature as coordinate curves. Equations for this special coordinate system will be established.

When coordinates coincide with the orthogonal lines of curvature, i.e., $x^1 = \alpha$ and $x^2 = \beta$, then equations (3-2-4) and (3-2-12) through (3-2-14) inclusive yield

$$a_{\alpha\beta} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} A_1^2 & 0 \\ 0 & A_2^2 \end{pmatrix} ,$$

$$b_{\alpha\beta} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} -\frac{A_1^2}{r_1} & 0 \\ 0 & -\frac{A_2^2}{r_2} \end{pmatrix} ,$$

$$a = A_1^2 A_2^2 \quad , \quad b = \frac{A_1^2 A_2}{r_1 r_2} \quad . \tag{3-4-1a-d}$$

The surface Christoffel symbols take the following form

$$\bar{\Gamma}_{111} = A_1 A_{1,\alpha} \equiv A_1 A_{1,1} \quad , \quad \bar{\Gamma}_{112} = -A_1 A_{1,\beta} \equiv -A_1 A_{1,2} \quad ,$$

$$\bar{\Gamma}_{122} = A_2 A_{2,1} \quad , \quad \bar{\Gamma}_{222} = A_2 A_{2,2} \quad , \tag{3-4-2a-d}$$

$$\bar{\Gamma}_{11}^1 = \frac{1}{A_1} A_{1,1} \quad , \quad \bar{\Gamma}_{12}^1 = \frac{1}{A_1} A_{1,2} \quad ,$$

$$\bar{\Gamma}_{21}^2 = \frac{1}{A_2} A_{2,1} \quad , \quad \bar{\Gamma}_{22}^1 = \frac{1}{A_2} A_{2,2} \quad . \tag{3-4-2e-h}$$

Appropriate substitution of the above quantities into equations (3-2-22) and (3-2-23) gives the simplified form of the equations of Codazzi and Gauss, as follows:

$$\left(\frac{A_1}{r_1}\right)_{,2} = \frac{1}{r_2} A_{1,2} \quad , \quad \left(\frac{A_2}{r_2}\right)_{,1} = \frac{1}{r_1} A_{2,1} \quad , \tag{3-4-3a,b}$$

$$\left(\frac{1}{A_1} A_{2,1}\right) + \left(\frac{1}{A_2} A_{1,2}\right) + \frac{A_1 A_2}{r_1 r_2} = 0 \quad . \tag{3-4-4}$$

The variation of strains along the z direction can be related to the surface strain tensor and change of curvature tensor in general curvilinear coordinates as

$$\epsilon = (\bar{\gamma}_{\alpha\beta} + \rho_{\alpha\beta} Z) \frac{dx^{\alpha}}{ds} \frac{dx^{\beta}}{ds} \quad . \tag{3-4-5}$$

These strain components, on the other hand, may be written in the selected coordinate system as

$$\begin{aligned} \epsilon &= (\epsilon_1 + X_1 z)\left(\frac{A_1 d\alpha}{ds}\right)^2 \\ &+ (\epsilon_2 + X_2 z)\left(\frac{A_2 d\beta}{ds}\right)^2 \\ &+ (X_{\epsilon} + 2 X_{12} z)\left(\frac{A_1 d\alpha}{ds}\right)\left(\frac{A_2 d\beta}{ds}\right) \quad . \end{aligned} \tag{3-4-6}$$

Comparison of equations (3-4-8) and (3-4-9) gives the following translation law

$$\bar{\gamma}_{11} = A_1^2 \epsilon_1 \quad , \quad \bar{\gamma}_{22} = A_2^2 \epsilon_2 \quad , \quad \bar{\gamma}_{12} = \frac{A_1 A_2}{2} \gamma_{12} ,$$

$$\rho_{11} = A_1^2 \chi_1 \quad , \quad \rho_{22} = A_2^2 \chi_2 \quad , \quad \rho_{12} = A_1 A_2 \chi_{12} . \qquad \text{(3-4-7a-f)}$$

In terms of physical components, the vectors $\underset{\sim}{n}_\alpha$ and $\underset{\sim}{m}_\alpha$ may also be expressed as

$$\underset{\sim}{n}_\alpha = N_{\alpha 1} \underset{\sim}{a}_1 (a_{11})^{-\frac{1}{2}} + N_{\alpha 2} \underset{\sim}{a}_2 (a_{22})^{-\frac{1}{2}} + Q_\alpha \underset{\sim}{a}_3 \quad , \qquad \text{(3-4-8)}$$

and

$$\underset{\sim}{m}_\alpha = M_{\alpha 1} \underset{\sim}{a}^2 (a^{22})^{-\frac{1}{2}} + M_{\alpha 2} \underset{\sim}{a}^1 (a^{11})^{-\frac{1}{2}} . \qquad \text{(3-4-9)}$$

If equations (3-3-16, 17) and (3-4-8, 9) are compared, one obtains

$$N_{\alpha\beta} = n^{\alpha\beta} \left(\frac{a_{\beta\beta}}{a^{\alpha\alpha}} \right)^{\frac{1}{2}} \quad ,$$

$$Q_\alpha = q^\alpha (a^{\alpha\alpha})^{\frac{1}{2}} \quad , \qquad \text{(3-4-10a-c)}$$

$$M_{\alpha\beta} = \begin{pmatrix} M_1 & M_{12} \\ & \\ M_{21} & M_2 \end{pmatrix} = \begin{pmatrix} m^{11} \left(\frac{a_{11}}{a^{11}} \right)^{\frac{1}{2}} & -m^{12} (a)^{\frac{1}{2}} \\ & \\ m^{21} (a)^{\frac{1}{2}} & -m^{22} \left(\frac{a_{22}}{a^{22}} \right)^{\frac{1}{2}} \end{pmatrix} .$$

After suitable substitution of equations (3-4-1,2) and (3-4-5) through (3-4-10) into equations (3-3-3) to (3-3-30) inclusive, one obtains the desired set of conventional shell equations:

A. Stress-Strain Relations

$$\sigma_1 = \frac{E}{1-\nu^2}\left[\bar{\epsilon}_1 + \nu\bar{\epsilon}_2 + z\,(\chi_1 + \nu\,\chi_2)\right] - \frac{\epsilon^T E}{1-\nu} \quad ,$$

$$\sigma_2 = \frac{E}{1-\nu^2}\left[\bar{\epsilon}_2 + \nu\bar{\epsilon}_1 + z\,(\chi_2 + \nu\,\chi_1)\right] - \frac{\epsilon^T E}{1-\nu} \quad , \qquad (3\text{-}4\text{-}11a\text{-}c)$$

$$\tau_{12} = G\,(\bar{\gamma}_{12} + 2\,z\,\chi_{12}),$$

where $\bar{\epsilon}$'s , $\bar{\gamma}$ are the middle-surface strains, and χ's are the changes of curvature and torsion.

B. Strain-Displacement Relations

$$\epsilon_1 = \bar{\epsilon}_1 + z\,\chi_1 \quad ,$$

$$\epsilon_2 = \bar{\epsilon}_2 + z\,\chi_2 \quad , \qquad (3\text{-}4\text{-}12a\text{-}c)$$

$$\gamma_{12} = \bar{\gamma}_{12} + 2\,z\,\chi_{12} \quad ,$$

where

$$\bar{\epsilon}_1 = \frac{1}{A_1}u_{1,1} + \frac{1}{A_1A_2}u_2A_{1,2} + \frac{u_3}{r_1} \quad ,$$

$$\bar{\epsilon}_2 = \frac{1}{A_2}u_{2,2} + \frac{1}{A_1A_2}u_1A_{2,1} + \frac{u_3}{r_2} \quad ,$$

$$\bar{\gamma}_{12} = \frac{A_2}{A_1}\left(\frac{u_2}{A_2}\right)_{,1} + \frac{A_1}{A_2}\left(\frac{u_1}{A_1}\right)_{,2} \quad ,$$

$$\chi_1 = \frac{1}{A_1}\omega_{1,1} + \frac{1}{A_1A_2}\omega_2 A_{1,2} \quad ,$$

$$\chi_2 = \frac{1}{A_2}\omega_{2,2} + \frac{1}{A_1A_2}\omega_1 A_{2,1} \quad ,$$

$$\begin{aligned} \chi_{12} = &-\frac{1}{A_1A_2}\left(u_{3,12} - \frac{1}{A_1} A_{1,2}u_{3,1} - \frac{1}{A_2} A_{2,1}u_{3,2}\right) \\ &+ \frac{1}{r_1}\left(\frac{1}{A_2}u_{1,2} - \frac{1}{A_1A_2}u_1 A_{1,2}\right) \\ &+ \frac{1}{r_2}\left(\frac{1}{A_1}u_{2,1} - \frac{1}{A_1A_2}u_2 A_{2,1}\right) \end{aligned} \qquad \text{(3-4-13a-f)}$$

in which the rotations $\omega_{(\ )}$ of the normal to the middle-surface in the direction α and β are respectively given by

$$\begin{aligned} \omega_1 &= \frac{u_1}{r_1} - \frac{u_{3,1}}{A_1} \quad , \\ \omega_2 &= \frac{u_2}{r_2} - \frac{u_{3,2}}{A_2} \quad . \end{aligned} \qquad \text{(3-4-14a,b)}$$

C. <u>Equilibrium Equations</u> (Fig. 3-2)

$$(A_2N_1)_{,1} + (A_1N_{21})_{,2} + N_{12}A_{1,2} - N_2A_{2,1} + A_1A_2\left(p_1 + \frac{Q_1}{r_1}\right) = 0 \quad ,$$

$$(A_2N_{12})_{,1} + (A_1N_2)_{,2} + N_{21}A_{2,1} - N_1A_{1,2} + A_1A_2\left(p_2 + \frac{Q_2}{r_2}\right) = 0 \quad ,$$

$$(A_2Q_1)_{,1} + (A_1Q_2)_{,2} + A_1A_2\left(p_3 - \frac{N_1}{r_1} - \frac{N_2}{r_2}\right) = 0 \quad ,$$

$$(A_2M_1)_{,1} + (A_1M_{21})_{,2} + M_{12}A_{1,2} - M_2A_{2,1} - A_1A_2Q_1 = 0 ,$$

$$(A_2M_{12})_{,1} + (A_1M_2)_{,2} + M_{21}A_{2,1} - M_1A_{1,2} - A_1A_2Q_2 = 0 , \quad (3\text{-}4\text{-}15a\text{-}f)$$

$$N_{12} - N_{21} + \frac{M_{12}}{r_1} - \frac{M_{21}}{r_2} = 0 .$$

D. <u>Constitutive Equations</u>

$$N_1 = \frac{Eh}{1-\nu^2}(\bar{\epsilon}_1 + \nu\bar{\epsilon}_2) + N^T ,$$

$$N_2 = \frac{Eh}{1-\nu^2}(\bar{\epsilon}_2 + \nu\bar{\epsilon}_1) + N^T ,$$

$$N_{12} = hG\left(\bar{\gamma} + \frac{h^2}{6r_2}\chi_{12}\right) , \qquad N_{21} = hG\left(\bar{\gamma} + \frac{h^2}{6r_1}\chi_{21}\right) , \quad (3\text{-}4\text{-}16a\text{-}i)$$

$$M_1 = D(\chi_1 + \nu\chi_2) + M^T ,$$

$$M_2 = D(\chi_2 + \nu\chi_1) + M^T ,$$

$$M_{12} = M_{21} = D(1-\nu)\chi_{12} ,$$

where

$$N^T = -\frac{E}{1-\nu}\int_{-\frac{h}{2}}^{\frac{h}{2}} \epsilon^T dz ,$$

$$M^T = -\frac{E}{1-\nu}\int_{-\frac{h}{2}}^{\frac{h}{2}} \epsilon^T z\, dz .$$

E. Rotation in Middle-Surface about the Normal

$$\Phi = \frac{1}{2}\left(\frac{1}{A_1} u_{2,1} - \frac{1}{A_2} u_{1,2} - \frac{u_1}{A_1 A_2} A_{1,2} + \frac{u_2}{A_1 A_2} A_{2,1}\right) \tag{3-4-17}$$

It should be pointed out that the effect of changes of temperature has been provided for, as shown in equations (3-4-11) and (3-4-16).

The sign conventions for stresses and deformations adopted for the present investigation are given in Fig. 3-2 and Fig. 3-3 respectively.

If the shears Q_1 and Q_2 are eliminated in equations (3-4-15) and the stress-resultants as well as moments are substituted by equations (3-4-16) which in turn can be expressed by equations (3-4-12) and (3-4-13), one then obtains three governing differential equations of equilibrium (3-4-18). The coefficients of these equations are shown in Table 3-4-1. It is noted that the Codazzi and Gauss equations (3-4-3) and (3-4-4) have been used in differentiations during the process of obtaining these governing equations.

$$a_1 u_{1,11} + a_2 u_{1,22} + a_3 u_{1,1} + a_4 u_{1,2} + a_5 u_1 + a_6 u_{2,12} + a_7 u_{2,1} + a_8 u_{2,2} + a_9 u_2 + a_{10} u_{3,111} + a_{11} u_{3,122}$$

$$+ a_{12} u_{3,11} + a_{13} u_{3,12} + a_{14} u_{3,22} + a_{15} u_{3,1} + a_{16} u_{3,2} + a_{17} u_3 = -\frac{h^2}{12D}\left(A_1 A_2 P_1 + A_2 N^T_{,1} + \frac{A_2}{r_1} M^T_{,1}\right)$$

$$b_1 u_{1,12} + b_2 u_{1,1} + b_3 u_{1,2} + b_4 u_1 + b_5 u_{2,11} + b_6 u_{2,22} + b_7 u_{2,1} + b_8 u_{2,2} + b_9 u_2 + b_{10} u_{3,112} + b_{11} u_{3,222}$$

$$+ b_{12} u_{3,11} + b_{13} u_{3,12} + b_{14} u_{3,22} + b_{15} u_{3,1} + b_{16} u_{3,2} + b_{17} u_3 = -\frac{h^2}{12D}\left(A_1 A_2 P_2 + A_1 N^T_{,2} + \frac{A_1}{r_2} M^T_{,2}\right)$$

(3-4-18a-c)

$$c_1 u_{1,111} + c_2 u_{1,122} + c_3 u_{1,11} + c_4 u_{1,12} + c_5 u_{1,22} + c_6 u_{1,1} + c_7 u_{1,2} + c_8 u_1 + c_9 u_{2,112} + c_{10} u_{2,222} + c_{11} u_{2,11} +$$

$$c_{12} u_{2,12} + c_{13} u_{2,22} + c_{14} u_{2,1} + c_{15} u_{2,2} + c_{16} u_2 + c_{17} u_{3,1111} + c_{18} u_{3,1122} + c_{19} u_{3,2222} + c_{20} u_{3,111} + c_{21} u_{3,112}$$

$$+ c_{22} u_{3,122} + c_{23} u_{3,222} + c_{24} u_{3,11} + c_{25} u_{3,12} + c_{26} u_{3,22} + c_{27} u_{3,1} + c_{28} u_{3,2} + c_{29} u_3 =$$

$$-A_1 A_2\left(P_3 - \frac{N^T}{r_1} - \frac{N^T}{r_2} + \frac{M^T_{,11}}{A_1^2} + \frac{A_{2,1}}{A_1^2 A_2} M^T_{,1} + \frac{A_{1,1} A_2}{A_1^2} M^T_{,1} + \frac{M^T_{,22}}{A_2^2} + \frac{A_{1,2}}{A_1 A_2^2} M^T_{,2} + \frac{A_{2,2} A_1}{A_2^2} M^T_{,2}\right)$$

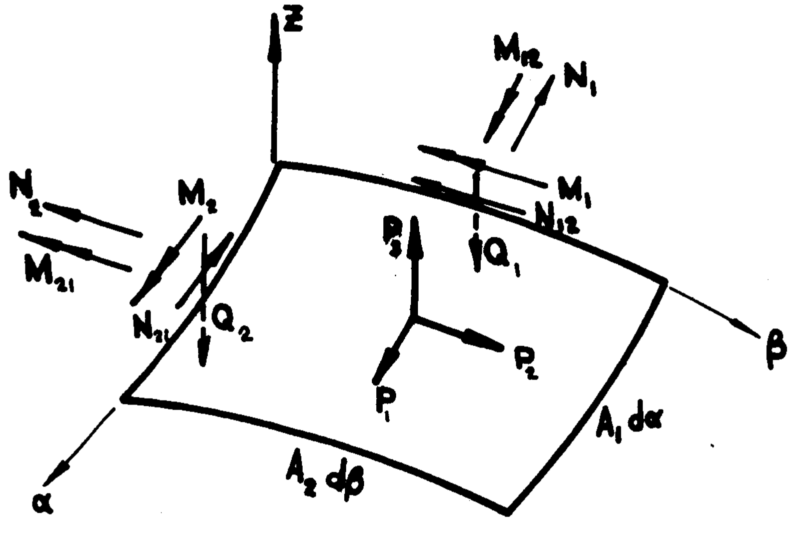


FIG. 3-2 STRESS RESULTANTS, MOMENTS AND LOADS

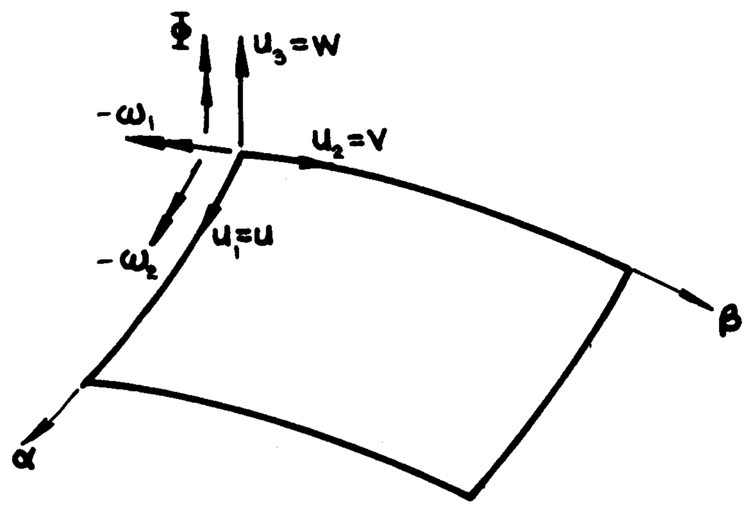


FIG. 3-3 DISPLACEMENTS AND ROTATIONS

Table 3-4-1 Coefficients a_i , b_i , and c_i of General Shell

$a_1=$	$\frac{A_2}{A_1}+\frac{h^2}{12}\left[\frac{A_2}{A_1 r_1^2}\right]$
$a_2=$	$\frac{(1-\nu)}{2}\frac{A_1}{A_2}+\frac{h^2}{12}\left[2\frac{A_1(1-\nu)}{A_2 r_1^2}\right]$
$a_3=$	$\frac{h^2}{D}\left(\frac{A_2 D}{A_1 h^2}\right)_{,1}+\frac{h^2}{12}\left[\frac{1}{D}\left(\frac{A_2 D}{A_1 r_1^2}\right)_{,1}\right]$
$a_4=$	$\frac{h^2}{D}\left(\frac{A_1}{A_2}\frac{D(1-\nu)}{2h^2}\right)_{,2}+\frac{h^2}{12}\left[\frac{2}{D}\left(\frac{A_1}{A_2 r_1^2}D(1-\nu)\right)_{,2}\right]$
$a_5=$	$A_2\frac{h^2}{D}\left(\frac{A_{2,1}}{A_1A_2}\frac{\nu D}{h^2}\right)_{,1}-\frac{h^2}{A_1D}\left(\frac{A_1A_{1,2}}{2h^2A_2}D(1-\nu)\right)_{,2}-(1-\nu)\frac{A_{2,1}^2}{A_1A_2}+\frac{h^2}{12}\left[\frac{A_2}{Dr_1}\left(\frac{D}{A_1}\left(\frac{1}{r_1}\right)_{,1}+\nu\frac{A_{2,1}}{A_2r_1}\right)_{,1}-\frac{2}{A_1D}\left(\frac{A_1A_{1,2}}{A_2r_1^2}D(1-\nu)\right)_{,2}-\frac{(1-\nu)}{A_1A_2r_1^2}\left(\frac{A_2}{r_2}\right)_{,1}(A_2r_1)_{,1}\right]$
$a_6=$	$\frac{1+\nu}{2}+\frac{h^2}{12}\left[\frac{2-\nu}{r_1r_2}\right]$
$a_7=$	$\frac{A_{1,12}}{A_1}+\frac{h^2}{A_1D}\left(\frac{A_1}{2h^2}D(1-\nu)\right)_{,12}+\frac{h^2}{12}\left[\frac{2}{A_1D}\left(\frac{A_1}{r_1r_2}D(1-\nu)\right)_{,12}+\frac{1}{r_1}\left(\frac{A_{1,22}}{A_1r_2}+\nu\left(\frac{1}{r_2}\right)_{,2}\right)\right]$
$a_8=$	$\frac{h^2}{D}\left(\nu\frac{D}{h^2}\right)_{,1}-\frac{3-\nu}{2}\frac{A_{2,1}}{A_2}+\frac{h^2}{12}\left[\frac{1}{r_1D}\left(\frac{\nu D}{r_2}\right)_{,1}-(3-2\nu)\frac{A_{2,1}}{A_2r_1r_2}\right]$
$a_9=$	$A_2\frac{h^2}{D}\left(\frac{A_{1,2}}{A_1A_2}\frac{D}{h^2}\right)_{,1}-\frac{h^2}{A_1D}\left(\frac{A_1A_{2,1}}{2h^2A_2}D(1-\nu)\right)_{,2}+(1-\nu)\frac{A_{1,2}A_{2,1}}{A_1A_2}+\frac{h^2}{12}\left[\frac{A_2}{Dr_1}\left(\frac{D}{A_2}\left\{\frac{A_{1,2}}{A_1r_2}+\nu\left(\frac{1}{r_2}\right)_{,2}\right\}\right)_{,1}-\frac{2}{A_1D}\left(\frac{A_1A_{2,1}}{A_2r_1r_2}D(1-\nu)\right)_{,2}-\frac{1-\nu}{A_1A_2r_2^2}\left(\frac{A_2}{r_2}\right)_{,1}(A_1r_2)_{,2}\right]$
$a_{10}=$	$-\frac{h^2}{12}\left[\frac{A_2}{A_1^2r_1}\right]$
$a_{11}=$	$-\frac{h^2}{12}\left[\frac{2-\nu}{A_2r_1}\right]$
$a_{12}=$	$-\frac{h^2}{12}\left[\frac{A_1}{r_1D}\left(\frac{A_2}{A_1^3}D\right)_{,1}\right]$
$a_{13}=$	$-\frac{h^2}{12}\left[\frac{1}{A_2r_1}\left\{\frac{A_{1,2}}{A_1}-\nu\frac{A_{2,2}}{A_2}\right\}+\frac{2}{D}\left(\frac{D(1-\nu)}{A_2r_1}\right)_{,2}\right]$
$a_{14}=$	$-\frac{h^2}{12}\left[\frac{1}{A_2r_1D}(\nu D)_{,1}+\frac{3-\nu}{r_1}\left(\frac{1}{A_2}\right)_{,1}\right]$
$a_{15}=$	$A_2\left(\frac{1}{r_1}+\frac{\nu}{r_2}\right)+\frac{h^2}{12}\left[\frac{A_2}{r_1D}\left(\frac{D}{A_1^2}\left\{\frac{A_{1,1}}{A_1}-\nu\frac{A_{2,1}}{A_2}\right\}\right)+\frac{2}{A_1D}\left(\frac{A_{2,1}}{A_2r_1}D(1-\nu)\right)_{,12}+\frac{(1-\nu)}{A_1^3A_2}\left(\frac{A_2}{r_2}\right)_{,1}(A_1A_2)_{,1}\right]$
$a_{16}=$	$\frac{h^2}{12}\left[-\frac{A_2}{r_1D}\left(\frac{D}{A_2^2}\left\{\frac{A_{1,12}}{A_1}-\nu\frac{A_{2,12}}{A_2}\right\}\right)_{,1}+\frac{2}{A_1D}\left(\frac{A_1A_{2,1}}{A_2^2r_1}D(1-\nu)\right)_{,12}-\frac{(1-\nu)}{A_1A_2^3}\left(\frac{A_2}{r_2}\right)_{,1}(A_1A_2)_{,2}\right]$
$a_{17}=$	$\frac{h^2}{A_2D}\left(A_2^2\frac{D}{h^2}\left\{\frac{1}{r_1}+\frac{\nu}{r_2}\right\}\right)-(1+\nu)A_{2,1}\left\{\frac{1}{r_1}+\frac{1}{r_2}\right\}$

Table 3-4-1 Coefficients a_i , b_i , and c_i of General Shell (Cont.)

$b_1=$	$\frac{1+\nu}{2}+\frac{h^2}{12}\left[\frac{2-\nu}{r_1r_2}\right]$
$b_2=$	$\frac{h^2}{D}\left(\frac{\nu D}{h^2}\right)_{,2}-\frac{3-\nu}{2}\frac{A_{1,2}}{A_1}+\frac{h^2}{12}\left[\frac{1}{r_2D}\left(\frac{\nu D}{r_1}\right)_{,2}-(3-2\nu)\frac{A_{1,2}}{A_1r_1r_2}\right]$
$b_3=$	$\frac{h^2}{A_2D}\left(\frac{A_2}{2h^2}D(1-\nu)\right)_{,1}+\frac{A_{2,1}}{A_2}+\frac{h^2}{12}\left[\frac{2}{A_2D}\left(\frac{A_2}{r_1r_2}D(1-\nu)\right)_{,1}+\frac{1}{r_2}\left(\frac{A_{2,1}}{A_2r_1}+\nu\left(\frac{1}{r_1}\right)_{,1}\right)\right]$
$b_4=$	$A_1\frac{h^2}{D}\left(\frac{A_{2,1}}{A_1A_2}\frac{D}{h^2}\right)_{,2}-\frac{h^2}{A_2D}\left(\frac{A_2A_{1,2}}{2h^2A_1}D(1-\nu)\right)_{,1}+(1-\nu)\frac{A_{1,2}A_{2,1}}{A_1A_2}+\frac{h^2}{12}\left[\frac{A_1}{r_2D}\left(\frac{D}{A_1}\left\{\frac{A_{2,1}}{A_2r_1}+\nu\left(\frac{1}{r_1}\right)_{,1}\right\}\right)_{,2}-\frac{2}{A_2D}\left(\frac{A_2A_{1,2}}{A_1r_1r_2}D(1-\nu)\right)_{,1}-\frac{(1-\nu)}{A_1A_2r_1^2}\left(\frac{A_1}{r_1}\right)_{,2}(A_2r_1)_{,2}\right]$
$b_5=$	$\frac{(1-\nu)}{2}\frac{A_2}{A_1}+\frac{h^2}{12}\left[2\frac{A_2(1-\nu)}{A_1r_2^2}\right]$
$b_6=$	$\frac{A_1}{A_2}+\frac{h^2}{12}\left[\frac{A_1}{A_2r_2^2}\right]$
$b_7=$	$\frac{h^2}{D}\left(\frac{A_2D(1-\nu)}{A_1\,2h^2}\right)_{,1}+\frac{h^2}{12}\left[\frac{2}{D}\left(\frac{A_2}{A_1r_2^2}D(1-\nu)\right)_{,1}\right]$
$b_8=$	$\frac{h^2}{D}\left(\frac{A_1D}{A_2h^2}\right)_{,2}+\frac{h^2}{12}\left[\frac{1}{D}\left(\frac{A_1D}{A_2r_2^2}\right)_{,2}\right]$
$b_9=$	$A_1\frac{h^2}{D}\left(\frac{A_{1,2}}{A_1A_2}\nu\frac{D}{h^2}\right)_{,2}-\frac{h^2}{A_2D}\left(\frac{A_2A_{2,1}}{2h^2A_1}D(1-\nu)\right)_{,1}-(1-\nu)\frac{A_{1,2}^2}{A_1A_2}+\frac{h^2}{12}\left[\frac{A_1}{r_2D}\left(\frac{D}{A_2}\left\{\left(\frac{1}{r_2}\right)_{,2}+\nu\frac{A_{1,2}}{A_1r_2}\right\}\right)_{,2}-\frac{2}{A_2D}\left(\frac{A_2A_{2,1}}{A_1r_2^2}D(1-\nu)\right)_{,1}-\frac{(1-\nu)}{A_1A_2r_2^2}\left(\frac{A_1}{r_1}\right)_{,2}(A_1r_2)_{,2}\right]$
$b_{10}=$	$-\frac{h^2}{12}\left[\frac{2-\nu}{A_1r_2}\right]$
$b_{11}=$	$-\frac{h^2}{12}\left[\frac{A_1}{A_2^2r_2}\right]$
$b_{12}=$	$-\frac{h^2}{12}\left[\frac{1}{A_1r_2D}(\nu D)_{,2}+\frac{3-\nu}{r_2}\left(\frac{1}{A_1}\right)_{,2}\right]$
$b_{13}=$	$-\frac{h^2}{12}\left[\frac{1}{A_1r_2}\left\{\frac{A_{2,1}}{A_2}-\nu\frac{A_{1,1}}{A_1}\right\}+\frac{2}{D}\left(\frac{D(1-\nu)}{A_1r_2}\right)_{,1}\right]$
$b_{14}=$	$-\frac{h^2}{12}\left[\frac{A_2}{r_2D}\left(\frac{A_1}{A_2^3}D\right)_{,2}\right]$
$b_{15}=$	$\frac{h^2}{12}\left[-\frac{A_1}{r_2D}\left(\frac{D}{A_1^2}\left\{\frac{A_{2,1}}{A_2}-\nu\frac{A_{1,1}}{A_1}\right\}\right)_{,2}+\frac{2}{A_2D}\left(\frac{A_2A_{1,2}}{A_1^2r_2}D(1-\nu)\right)_{,1}-\frac{(1-\nu)}{A_1^3A_2}\left(\frac{A_1}{r_1}\right)_{,2}(A_1A_2)_{,1}\right]$
$b_{16}=$	$A_1\left(\frac{1}{r_2}+\frac{\nu}{r_1}\right)+\frac{h^2}{12}\left[\frac{A_1}{r_2D}\left(\frac{D}{A_2^2}\left\{\frac{A_{2,2}}{A_2}-\nu\frac{A_{1,2}}{A_1}\right\}\right)_{,2}+\frac{2}{A_2D}\left(\frac{A_{2,1}}{A_1r_2}D(1-\nu)\right)_{,1}+\frac{1-\nu}{A_1A_2^3}\left(\frac{A_1}{r_1}\right)_{,2}(A_1A_2)_{,2}\right]$
$b_{17}=$	$\frac{h^2}{A_1D}\left(A_1^2\frac{D}{h^2}\left\{\frac{1}{r_2}+\frac{\nu}{r_1}\right\}\right)_{,2}-(1+\nu)A_{1,2}\left\{\frac{1}{r_2}+\frac{1}{r_1}\right\}$

Table 3-4-1 Coefficients a_i , b_i , and c_i of General Shell (Cont.)

$c_1 =$	$\dfrac{A_2 D}{A_1^2 r_1}$
$c_2 =$	$(2-\nu)\dfrac{D}{A_2 r_1}$
$c_3 =$	$\dfrac{D}{A_1 A_2}\left(\dfrac{A_2^2}{A_1 r_1}\right)_{,y_1} + 2\dfrac{A_2}{A_1}\left(\dfrac{D}{A_1 r_1}\right)_{,y_1}$
$c_4 =$	$\left(\dfrac{A_1}{A_2}\right)_{,y_2}\dfrac{\nu D}{A_1 r_1} + 2\dfrac{A_1}{A_2}\left(\dfrac{\nu D}{A_1 r_1}\right)_{,y_2} + \left(\dfrac{1}{A_1}\right)_{,y_2}\left(\dfrac{A_1 D(1-\nu)}{A_2 r_1}\right) + 2\left(\dfrac{D(1-\nu)}{A_2 r_1}\right)_{,y_2}$
$c_5 =$	$\dfrac{A_1}{A_2}\left(\dfrac{D}{A_1}\left\{\dfrac{A_{2,y_1}}{A_2 r_1} + \nu\left(\dfrac{1}{r_1}\right)_{,y_1}\right\}\right) + \dfrac{2}{A_2}\left(\dfrac{D(1-\nu)}{r_1}\right)_{,y_1}$
$c_6 =$	$\left(\dfrac{A_2}{A_1}\right)_{,y_1}\left(\dfrac{D}{A_1}\left\{\left(\dfrac{1}{r_1}\right)_{,y_1} + \nu\dfrac{A_{2,y_1}}{A_2 r_1}\right\}\right) + 2\dfrac{A_2}{A_1}\left(\dfrac{D}{A_1}\left\{\left(\dfrac{1}{r_1}\right)_{,y_1} + \nu\dfrac{A_{2,y_1}}{A_2 r_1}\right\}\right)_{,y_1} - \dfrac{1}{A_1^2}\left(\dfrac{A_{1,y_2}A_2}{A_2 r_1}D(1-\nu)\right)_{,y_2} - 2\left(\dfrac{A_{1,y_2}D(1-\nu)}{A_1 A_2 r_1}\right)_{,y_2} + \left(\dfrac{A_2 A_{2,y_1}}{A_1^2}D(1-\nu)\right)_{,y_1}\dfrac{1}{A_2 r_1}$ $+2\dfrac{D(1-\nu)A_{2,y_1}A_2}{A_1^2}\left(\dfrac{1}{A_2 r_1}\right)_{,y_1} + \left[\dfrac{A_2}{A_1}\left(\dfrac{D}{A_1 r_1}\right)_{,y_1}\right]_{,y_1} + \left[\dfrac{A_1}{A_2}\left(\dfrac{\nu D}{A_1 r_1}\right)_{,y_2}\right]_{,y_2} - 12\dfrac{D}{h^2 r_1}A_2\left(1+\nu\dfrac{r_1}{r_2}\right)$
$c_7 =$	$\left(\dfrac{A_1}{A_2}\right)_{,y_2}\left(\dfrac{D}{A_1}\left\{\dfrac{A_{2,y_1}}{A_2 r_1} + \nu\left(\dfrac{1}{r_1}\right)_{,y_1}\right\}\right) + 2\dfrac{A_1}{A_2}\left(\dfrac{D}{A_1}\left\{\dfrac{A_{2,y_1}}{A_2 r_1} + \nu\left(\dfrac{1}{r_1}\right)_{,y_1}\right\}\right)_{,y_2} - \dfrac{1}{A_2^2}\left(\dfrac{A_{1,y_2}A_2}{A_1 r_1}D(1-\nu)\right)_{,y_1} + \left[\dfrac{1}{A_1^2}\left(\dfrac{A_1^2 D(1-\nu)}{A_2 r_1}\right)_{,y_2}\right]_{,y_1} + \left[\dfrac{1}{A_2^2}\left(\dfrac{A_2^2 D(1-\nu)}{A_2 r_1}\right)_{,y_1}\right]_{,y_2} - \left(\dfrac{A_{1,y_2}}{A_1}D(1-\nu)\right)_{,y_1}\dfrac{1}{A_2 r_1} - 2\dfrac{A_{1,y_2}}{A_1}D(1-\nu)\left(\dfrac{1}{A_2 r_1}\right)_{,y_1}$
$c_8 =$	$\left[\dfrac{A_2}{A_1}\left(\dfrac{D}{A_1}\left\{\left(\dfrac{1}{r_1}\right)_{,y_1} + \nu\dfrac{A_{2,y_1}}{A_2 r_1}\right\}\right)_{,y_1}\right]_{,y_1} + \left[\dfrac{A_1}{A_2}\left(\dfrac{D}{A_1}\left\{\dfrac{A_{2,y_1}}{A_2 r_1} + \nu\left(\dfrac{1}{r_1}\right)_{,y_1}\right\}\right)_{,y_2}\right]_{,y_2} - \left[\dfrac{1}{A_1^2}\left(\dfrac{A_{1,y_2}A_1}{A_2 r_1}D(1-\nu)\right)_{,y_2}\right]_{,y_1} - \left[\dfrac{1}{A_2^2}\left(\dfrac{A_{1,y_2}A_2}{A_1 r_1}D(1-\nu)\right)_{,y_1}\right]_{,y_2} - \left[\dfrac{A_{1,y_2}}{A_1}\left(\dfrac{1}{A_2 r_1}\right)_{,y_1}D(1-\nu)\right]_{,y_2} + \left[\dfrac{A_{2,y_1}A_2}{A_1^2}\left(\dfrac{1}{A_2 r_1}\right)_{,y_1}D(1-\nu)\right]_{,y_1} - 12\dfrac{DA_{2,y_1}}{h^2 r_2}\left(1+\nu\dfrac{r_2}{r_1}\right)$
$c_9 =$	$(2-\nu)\dfrac{D}{A_1 r_2}$
$c_{10} =$	$\dfrac{A_1 D}{A_2^2 r_2}$
$c_{11} =$	$\dfrac{A_2}{A_1}\left(\dfrac{D}{A_2}\left\{\dfrac{A_{1,y_2}}{A_1 r_2} + \nu\left(\dfrac{1}{r_2}\right)_{,y_2}\right\}\right) + \dfrac{2}{A_1}\left(\dfrac{D(1-\nu)}{r_2}\right)_{,y_2}$
$c_{12} =$	$\left(\dfrac{A_2}{A_1}\right)_{,y_1}\dfrac{\nu D}{A_2 r_2} + 2\dfrac{A_2}{A_1}\left(\dfrac{\nu D}{A_2 r_2}\right)_{,y_1} + \left(\dfrac{1}{A_2}\right)_{,y_1}\left(\dfrac{A_2 D(1-\nu)}{A_1 r_2}\right) + 2\left(\dfrac{D(1-\nu)}{A_1 r_2}\right)_{,y_1}$
$c_{13} =$	$\dfrac{D}{A_1 A_2}\left(\dfrac{A_1^2}{A_2 r_2}\right)_{,y_2} + 2\dfrac{A_1}{A_2}\left(\dfrac{D}{A_2 r_2}\right)_{,y_2}$
$c_{14} =$	$\left(\dfrac{A_2}{A_1}\right)_{,y_1}\left(\dfrac{D}{A_2}\left\{\dfrac{A_{1,y_2}}{A_1 r_2} + \nu\left(\dfrac{1}{r_2}\right)_{,y_2}\right\}\right) + 2\dfrac{A_2}{A_1}\left(\dfrac{D}{A_2}\left\{\dfrac{A_{1,y_2}}{A_1 r_2} + \nu\left(\dfrac{1}{r_2}\right)_{,y_2}\right\}\right)_{,y_1} - \dfrac{1}{A_1^2}\left(\dfrac{A_{2,y_1}A_1}{A_2 r_2}D(1-\nu)\right)_{,y_2} + \left[\dfrac{1}{A_1^2}\left(\dfrac{A_1^2 D(1-\nu)}{A_1 r_2}\right)_{,y_2}\right]_{,y_1} + \left[\dfrac{1}{A_2^2}\left(\dfrac{A_2^2 D(1-\nu)}{A_1 r_2}\right)_{,y_1}\right]_{,y_2} - \left(\dfrac{A_{2,y_1}}{A_2}D(1-\nu)\right)_{,y_2}\dfrac{1}{A_1 r_2} - 2\dfrac{A_{2,y_1}}{A_2}D(1-\nu)\left(\dfrac{1}{A_1 r_2}\right)_{,y_2}$
$c_{15} =$	$\left(\dfrac{A_1}{A_2}\right)_{,y_2}\left(\dfrac{D}{A_2}\left\{\left(\dfrac{1}{r_2}\right)_{,y_2} + \nu\dfrac{A_{1,y_2}}{A_1 r_2}\right\}\right) + 2\dfrac{A_1}{A_2}\left(\dfrac{D}{A_2}\left\{\left(\dfrac{1}{r_2}\right)_{,y_2} + \nu\dfrac{A_{1,y_2}}{A_1 r_2}\right\}\right)_{,y_2} - \dfrac{1}{A_2^2}\left(\dfrac{A_{2,y_1}A_2}{A_1 r_2}D(1-\nu)\right)_{,y_1} - 2\left(\dfrac{A_{2,y_1}D(1-\nu)}{A_1 A_2 r_2}\right)_{,y_1} + \left(\dfrac{A_{1,y_2}A_1}{A_2^2}D(1-\nu)\right)_{,y_2}\dfrac{1}{A_1 r_2}$ $+2\dfrac{D(1-\nu)A_{1,y_2}A_1}{A_2^2}\left(\dfrac{1}{A_1 r_2}\right)_{,y_2} + \left[\dfrac{A_1}{A_2}\left(\dfrac{D}{A_2 r_2}\right)_{,y_2}\right]_{,y_2} + \left[\dfrac{A_2}{A_1}\left(\dfrac{\nu D}{A_2 r_2}\right)_{,y_1}\right]_{,y_1} - 12\dfrac{D}{h^2 r_2}A_1\left(1+\nu\dfrac{r_2}{r_1}\right)$

Table 3-4-1 Coefficients a_i, b_i, and c_i of General Shell (Cont.)

$c_{16}=$	$\left[\frac{A_2}{A_1}\left(\frac{D}{A_2}\left\{\frac{A_{1,2}}{A_1 r_2}+\nu\left(\frac{1}{r_1}\right)_{,2}\right\}\right)_{,1}\right]_{,1}+\left[\frac{A_1}{A_2}\left(\frac{D}{A_2}\left\{\left(\frac{1}{r_2}\right)_{,2}+\nu\frac{A_{1,2}}{A_1 r_2}\right\}\right)_{,2}\right]_{,2}-\left[\frac{1}{A_1^2}\left(\frac{A_{2,1}A_1}{A_2 r_2}D(1-\nu)\right)_{,1}\right]_{,1}-\left[\frac{1}{A_2^2}\left(\frac{A_{2,1}A_2}{A_1 r_2}D(1-\nu)\right)_{,1}\right]_{,2}-\left[\frac{A_{2,1}}{A_2}\left(\frac{1}{A_1 r_2}\right)_{,2}D(1-\nu)+\left[\frac{A_{1,2}A_1}{A_2^2}\left(\frac{1}{A_1 r_2}\right)_{,2}D(1-\nu)\right]_{,1}\right]_{,2}-12\frac{DA_{1,2}}{h^2 r_1}\left(1+\nu\frac{r_1}{r_2}\right)$
$c_{17}=$	$-\left(\frac{A_2 D}{A_1^3}\right)$
$c_{18}=$	$-2\frac{D}{A_1 A_2}$
$c_{19}=$	$-\left(\frac{A_1 D}{A_2^3}\right)$
$c_{20}=$	$-2\left(\frac{A_2 D}{A_1^3}\right)_{,1}$
$c_{21}=$	$-\frac{A_2}{A_1}\left(\frac{D}{A_2^2}\left\{\frac{A_{1,22}}{A_1}-\nu\frac{A_{2,12}}{A_2}\right\}\right)-\frac{1}{A_1^2}\left(\frac{A_1 D(1-\nu)}{A_2}\right)_{,22}-\frac{A_1}{A_2}\left(\frac{\nu D}{A_1^2}\right)_{,22}-\left(\frac{D}{A_1 A_2}\right)_{,22}+3\frac{A_{1,22}}{A_1^2 A_2}D(1-\nu)$
$c_{22}=$	$-\frac{A_1}{A_2}\left(\frac{D}{A_1^2}\left\{\frac{A_{2,11}}{A_2}-\nu\frac{A_{1,11}}{A_1}\right\}\right)-\frac{1}{A_2^2}\left(\frac{A_2 D(1-\nu)}{A_1}\right)_{,11}-\frac{A_2}{A_1}\left(\frac{\nu D}{A_2^2}\right)_{,11}-\left(\frac{D}{A_1 A_2}\right)_{,11}+3\frac{A_{2,11}}{A_1 A_2^2}D(1-\nu)$
$c_{23}=$	$-2\left(\frac{A_1 D}{A_2^3}\right)_{,2}$
$c_{24}=$	$\left(\frac{A_2}{A_1}\right)_{,1}\left(\frac{D}{A_1^2}\left\{\frac{A_{1,11}}{A_1}-\nu\frac{A_{2,11}}{A_2}\right\}\right)+2\frac{A_2}{A_1}\left(\frac{D}{A_1^2}\left\{\frac{A_{1,11}}{A_1}-\nu\frac{A_{2,11}}{A_2}\right\}\right)_{,1}+\frac{1}{A_1^2}\left(\frac{A_{1,2}}{A_2}D(1-\nu)\right)_{,2}+2\left(\frac{A_{1,2}D(1-\nu)}{A_1^2 A_2}\right)_{,2}-\left(\frac{A_{2,1}}{A_1^4 A_2}D(1-\nu)\right)_{,1}A_1A_2-\left[\frac{A_1}{A_2}\left(\frac{\nu D}{A_1^2}\right)_{,2}\right]_{,2}-\left[\frac{A_2}{A_1}\left(\frac{D}{A_1^2}\right)_{,1}\right]_{,1}$
$c_{25}=$	$-\left(\frac{A_2}{A_1}\right)_{,1}\left(\frac{D}{A_2^2}\left\{\frac{A_{1,12}}{A_1}-\nu\frac{A_{2,12}}{A_2}\right\}\right)-2\frac{A_2}{A_1}\left(\frac{D}{A_2^2}\left\{\frac{A_{1,12}}{A_1}-\nu\frac{A_{2,12}}{A_2}\right\}\right)_{,1}-\left(\frac{A_1}{A_2}\right)_{,2}\left(\frac{D}{A_1^2}\left\{\frac{A_{2,11}}{A_2}-\nu\frac{A_{1,11}}{A_1}\right\}\right)-2\frac{A_1}{A_2}\left(\frac{D}{A_1^2}\left\{\frac{A_{2,11}}{A_2}-\nu\frac{A_{1,11}}{A_1}\right\}\right)_{,2}+\frac{A_1}{A_2}\left(\frac{A_{1,2}D(1-\nu)}{A_1^3}\right)_{,1}$ $+\frac{A_2}{A_1}\left(\frac{A_{2,1}D(1-\nu)}{A_2^3}\right)_{,2}-\left[\frac{1}{A_1}\left(\frac{D(1-\nu)}{A_2}\right)_{,2}\right]_{,1}-\left[\frac{1}{A_2}\left(\frac{D(1-\nu)}{A_1}\right)_{,1}\right]_{,2}$
$c_{26}=$	$\left(\frac{A_1}{A_2}\right)_{,2}\left(\frac{D}{A_2^2}\left\{\frac{A_{2,22}}{A_2}-\nu\frac{A_{1,22}}{A_1}\right\}\right)+2\frac{A_1}{A_2}\left(\frac{D}{A_2^2}\left\{\frac{A_{2,22}}{A_2}-\nu\frac{A_{1,22}}{A_1}\right\}\right)_{,2}+\frac{1}{A_2^2}\left(\frac{A_{2,1}}{A_1}D(1-\nu)\right)_{,1}+2\left(\frac{A_{2,1}D(1-\nu)}{A_1 A_2^2}\right)_{,1}-\left(\frac{A_{1,2}}{A_1 A_2^4}D(1-\nu)\right)_{,2}A_1A_2-\left[\frac{A_2}{A_1}\left(\frac{\nu D}{A_2^2}\right)_{,1}\right]_{,1}-\left[\frac{A_1}{A_2}\left(\frac{D}{A_2^2}\right)_{,2}\right]_{,2}$
$c_{27}=$	$\left[\frac{A_2}{A_1}\left(\frac{D}{A_1^2}\left\{\frac{A_{1,11}}{A_1}-\nu\frac{A_{2,11}}{A_2}\right\}\right)_{,1}\right]_{,1}-\left[\frac{A_1}{A_2}\left(\frac{D}{A_1^2}\left\{\frac{A_{2,11}}{A_2}-\nu\frac{A_{1,12}}{A_1}\right\}\right)_{,2}\right]_{,2}+\left[\frac{1}{A_1^2}\left(\frac{A_{1,2}A_2}{A_2^2}D(1-\nu)\right)_{,2}\right]_{,1}+\left[\frac{1}{A_2^2}\left(\frac{A_2A_1}{A_1^2}D(1-\nu)\right)_{,1}\right]_{,2}+\left[\frac{A_{2,1}}{A_1}\left(\frac{1}{A_1A_2}\right)_{,1}D(1-\nu)\right]_{,2}-\left[\frac{A_{2,1}A_2}{A_1^2}\left(\frac{1}{A_1A_2}\right)_{,1}D(1-\nu)\right]_{,2}$
$c_{28}=$	$-\left[\frac{A_2}{A_1}\left(\frac{D}{A_2^2}\left\{\frac{A_{1,12}}{A_1}-\nu\frac{A_{2,12}}{A_2}\right\}\right)_{,1}\right]_{,1}-\left[\frac{A_1}{A_2}\left(\frac{D}{A_2^2}\left\{\frac{A_{2,12}}{A_2}-\nu\frac{A_{1,12}}{A_1}\right\}\right)_{,2}\right]_{,2}+\left[\frac{1}{A_1^2}\left(\frac{A_{2,1}A_1}{A_2^2}D(1-\nu)\right)_{,2}\right]_{,1}+\left[\frac{1}{A_2^2}\left(\frac{A_{2,1}A_1}{A_1^2}D(1-\nu)\right)_{,1}\right]_{,2}+\left[\frac{A_{2,1}}{A_2}\left(\frac{1}{A_1A_2}\right)_{,2}D(1-\nu)\right]_{,1}-\left[\frac{A_{1,2}A_1}{A_2^2}\left(\frac{1}{A_1A_2}\right)_{,2}D(1-\nu)\right]_{,1}$
$c_{29}=$	$-12A_1A_2\frac{D}{h^2}\left(\frac{r_1^2+2\nu r_1 r_2+r_2^2}{r_1^2 r_2^2}\right)$

3.5 Differential Equations of Equilibrium for Specific Shells

Although the expressions for the coefficients of the governing differential equations (3-4-18) as listed in Table 3-4-1 and Table 3-4-2 are perfectly general for all classical shell elements, in practice it is more desirable to reduce these expressions to specific cases so that they can be readily applied in the numerical analysis. This is accomplished if the Lamé parameters and the principal radii of curvature of the specific shells such as those listed in Table 3-5-1 are introduced.

Table 3-5-1 PARAMETERS OF SPECIFIC SHELLS

SHELL	α	β	A_1	A_2	r_1	r_2
Sphere	ϕ	θ	R	$R\sin\phi$	R	R
Cone	x	θ	1	$x\sin\phi_3$	∞	$x\tan\phi_3$
Cylinder	x	θ	1	R	∞	R
Plate	x	Y	1	1	∞	∞

Appropriate substitution of these parameters into the general expressions for the coefficients of governing differential equations yields the specific coefficients for the spheres, cones, cylinders and plates as shown in Table 3-5-2.

Thus far we have considered that the modulus of elasticity (E), Poisson's ratio (ν) and the thickness of shell (h) are variables. If the material properties are constants and the variations of shell thickness are prescribed,

Table 3-5-2 Coefficients a_i , b_i , and c_i of Specific Shells

	SPHERE	CONE	CYLINDER	PLATE
$a_1 =$	$\sin\phi(1+k)$	$x\sin\phi_3$	R	1
$a_2 =$	$\frac{1-\nu}{2\sin\phi}(1+4k)$	$\frac{1-\nu}{2x\sin\phi_3}$	$\frac{1-\nu}{2R}$	$\frac{1-\nu}{2}$
$a_3 =$	$\cos\phi(1+k)+\frac{\sin\phi}{D}\left[h^2\left(\frac{D}{h^2}\right)_{,\phi}+kD_{,\phi}\right]$	$\sin\phi_3\left[1+x\frac{h^2}{D}\left(\frac{D}{h^2}\right)_{,x}\right]$	$R\frac{h^2}{D}\left(\frac{D}{h^2}\right)_{,x}$	$\frac{h^2}{D}\left(\frac{D}{h^2}\right)_{,x}$
$a_4 =$	$\frac{1}{2\sin\phi D}\left[h^2\left(\frac{D(1-\nu)}{h^2}\right)_{,\theta}+4k\left(D(1-\nu)\right)_{,\theta}\right]$	$\frac{h^2}{2x\sin\phi_3 D}\left(\frac{D(1-\nu)}{h^2}\right)_{,\theta}$	$\frac{h^2}{2RD}\left(\frac{D(1-\nu)}{h^2}\right)_{,\theta}$	$\frac{h^2}{2D}\left(\frac{D(1-\nu)}{h^2}\right)_{,y}$
$a_5 =$	$\frac{\cos\phi}{D}\left[h^2\left(\frac{\nu D}{h^2}\right)_{,\phi}+k(\nu D)_{,\phi}\right]-\frac{\cos^2\phi+\nu\sin^2\phi}{\sin\phi}(1+k)$	$-\frac{\sin\phi_3}{x}\left[1-x\frac{h^2}{D}\left(\frac{\nu D}{h^2}\right)_{,x}\right]$	0	0
$a_6 =$	$\frac{1+\nu}{2}+(2-\nu)k$	$\frac{1+\nu}{2}$	$\frac{1+\nu}{2}$	$\frac{1+\nu}{2}$
$a_7 =$	$\frac{1}{2D}\left[h^2\left(\frac{D(1-\nu)}{h^2}\right)_{,\theta}+4k\left(D(1-\nu)\right)_{,\theta}\right]$	$\frac{h^2}{2D}\left(\frac{D(1-\nu)}{h^2}\right)_{,\theta}$	$\frac{h^2}{2D}\left(\frac{D(1-\nu)}{h^2}\right)_{,\theta}$	$\frac{h^2}{2D}\left(\frac{D(1-\nu)}{h^2}\right)_{,y}$
$a_8 =$	$-\frac{\cot}{2}\left[3-\nu+2k(3-2\nu)\right]+\frac{1}{D}\left[h^2\left(\frac{\nu D}{h^2}\right)_{,\phi}+k(\nu D)_{,\phi}\right]$	$-\frac{3-\nu}{2x}+\frac{h^2}{D}\left(\frac{\nu D}{h^2}\right)_{,x}$	$\frac{h^2}{D}\left(\frac{\nu D}{h^2}\right)_{,x}$	$\frac{h^2}{D}\left(\frac{\nu D}{h^2}\right)_{,x}$
$a_9 =$	$-\frac{\cot\phi}{2D}\left[h^2\left(\frac{D(1-\nu)}{h^2}\right)_{,\theta}+4k\left(D(1-\nu)\right)_{,\theta}\right]$	$-\frac{h^2}{2Dx}\left(\frac{D(1-\nu)}{h^2}\right)_{,\theta}$	0	0
$a_{10} =$	$-k\sin\phi$	0	0	0
$a_{11} =$	$-(2-\nu)\frac{k}{\sin\phi}$	0	0	0
$a_{12} =$	$-k\left[\cos\phi+\sin\phi\frac{D_{,\phi}}{D}\right]$	0	0	0
$a_{13} =$	$-\frac{2k}{\sin\phi D}\left(D(1-\nu)\right)_{,\theta}$	0	0	0
$a_{14} =$	$\frac{k}{\sin\phi}\left[(3-\nu)\cot\phi-\frac{(\nu D)_{,\phi}}{D}\right]$	0	0	0
$a_{15} =$	$(1+\nu)\sin\phi+k\left[\frac{\cos^2\phi+\nu\sin^2\phi}{\sin\phi}-\cos\phi\frac{(\nu D)_{,\phi}}{D}\right]$	$\nu\cos\phi_3$	ν	0
$a_{16} =$	$2\frac{\cos\phi k}{\sin^2\phi D}\left(D(1-\nu)\right)_{,\theta}$	0	0	0
$a_{17} =$	$\sin\phi\frac{h^2}{D}\left(\frac{D(1+\nu)}{h^2}\right)_{,\phi}$	$-\frac{\cos\phi_3}{x}\left[1-x\frac{h^2}{D}\left(\frac{\nu D}{h^2}\right)_{,x}\right]$	$\frac{h^2}{D}\left(\frac{\nu D}{h^2}\right)_{,x}$	0

Table 3-5-2 Coefficients a_i , b_i , and c_i of Specific Shells (Cont.)

	SPHERE	CONE	CYLINDER	PLATE
$b_1=$	$\frac{1+\nu}{2}+(2-\nu)k$	$\frac{1+\nu}{2}$	$\frac{1+\nu}{2}$	$\frac{1+\nu}{2}$
$b_2=$	$\frac{\bar h^2}{D}\left(\frac{\nu D}{\bar h^2}\right)_{,\theta}+k\frac{(\nu D)_{,\theta}}{D}$	$\frac{\bar h^2}{D}\left(\frac{\nu D}{\bar h^2}\right)_{,\theta}$	$\frac{\bar h^2}{D}\left(\frac{\nu D}{\bar h^2}\right)_{,\theta}$	$\frac{\bar h^2}{D}\left(\frac{\nu D}{\bar h^2}\right)_{,y}$
$b_3=$	$\cot\phi\left[\frac{3-\nu}{2}+(3-2\nu)k\right]+\frac{1}{2D}\left[\bar h^2\left(\frac{D(1-\nu)}{\bar h^2}\right)_{,\phi}+4k(D(1-\nu))_{,\phi}\right]$	$\frac{1}{2x}\left[3-\nu+x\frac{\bar h^2}{D}\left(\frac{D(1-\nu)}{\bar h^2}\right)_{,x}\right]$	$\frac{\bar h^2}{2D}\left(\frac{D(1-\nu)}{\bar h^2}\right)_{,x}$	$\frac{\bar h^2}{2D}\left(\frac{D(1-\nu)}{\bar h^2}\right)_{,x}$
$b_4=$	$\frac{\cot\phi}{D}\left[\bar h^2\left(\frac{D}{\bar h^2}\right)_{,\theta}+kD_{,\theta}\right]$	$\frac{\bar h^2}{xD}\left(\frac{D}{\bar h^2}\right)_{,\theta}$	0	0
$b_5=$	$\sin\phi\frac{1-\nu}{2}\left[1+4k\right]$	$x\sin\phi_3\frac{1-\nu}{2}\left[1+\frac{4k}{x^2}\right]$	$\frac{(1-\nu)R}{2}\left[1+4k\right]$	$\frac{1-\nu}{2}$
$b_6=$	$\frac{1}{\sin\phi}\left[1+k\right]$	$\frac{1}{x\sin\phi_3}\left[1+\frac{k}{x^2}\right]$	$\frac{1}{R}\left[1+k\right]$	1
$b_7=$	$\cos\phi\frac{(1-\nu)}{2}(1+4k)+\frac{\sin\phi}{2D}\left[\bar h^2\left(\frac{D(1-\nu)}{\bar h^2}\right)_{,\phi}+4k(D(1-\nu))_{,\phi}\right]$	$\sin\phi_3\frac{1-\nu}{2}\left[1+\frac{\bar h^2 x}{D(1-\nu)}\left(\frac{D(1-\nu)}{\bar h^2}\right)_{,x}+\frac{4k}{x^2}+\frac{4k(D(1-\nu))_{,x}}{xD(1-\nu)}\right]$	$\frac{R}{2D}\left[\bar h^2\left(\frac{D(1-\nu)}{\bar h^2}\right)_{,x}+4k(D(1-\nu))_{,x}\right]$	$\frac{\bar h^2}{D}\left(\frac{D(1-\nu)}{2\bar h^2}\right)_{,y}$
$b_8=$	$\frac{1}{\sin\phi D}\left[\bar h^2\left(\frac{D}{\bar h^2}\right)_{,\theta}+kD_{,\theta}\right]$	$\frac{1}{x\sin\phi_3}\left[\frac{\bar h^2}{D}\left(\frac{D}{\bar h^2}\right)_{,\theta}+\frac{k}{Dx^2}D_{,\theta}\right]$	$\frac{1}{RD}\left[\bar h^2\left(\frac{D}{\bar h^2}\right)_{,\theta}+kD_{,\theta}\right]$	$\frac{\bar h^2}{D}\left(\frac{D}{\bar h^2}\right)_{,y}$
$b_9=$	$\frac{1-\nu}{2}\left(\frac{\sin^2\phi-\cos^2\phi}{\sin\phi}\right)(1+4k)-\frac{\cos\phi}{2D}\left[\bar h^2\left(\frac{D(1-\nu)}{\bar h^2}\right)_{,\phi}+4k(D(1-\nu))_{,\phi}\right]$	$-\frac{\sin\phi_3(1-\nu)}{2x}\left[1+\frac{\bar h^2 x}{D(1-\nu)}\left(\frac{D(1-\nu)}{\bar h^2}\right)_{,x}+\frac{4k}{x^2}+\frac{4k(D(1-\nu))_{,x}}{xD(1-\nu)}\right]$	0	0
$b_{10}=$	$-(2-\nu)k$	$-(2-\nu)k\frac{\tan\phi_3}{x}$	$-(2-\nu)kR$	0
$b_{11}=$	$-\frac{k}{\sin^2\phi}$	$-\frac{k}{x^3\sin\phi_3\cos\phi_3}$	$-\frac{k}{R}$	0
$b_{12}=$	$-k\frac{(\nu D)_{,\theta}}{D}$	$-\frac{k\tan\phi_3}{xD}(\nu D)_{,\theta}$	$-kR\frac{(\nu D)_{,\theta}}{D}$	0
$b_{13}=$	$-k\left[\cot\phi+2\frac{(D(1-\nu))_{,\phi}}{D}\right]$	$\frac{k\tan\phi_3}{x^2}\left[1-2\nu-\frac{2x}{D}(D(1-\nu))_{,x}\right]$	$-2kR(D(1-\nu))_{,x}$	0
$b_{14}=$	$-\frac{kD_{,\theta}}{\sin^2\phi D}$	$-\frac{kD_{,\theta}}{x^3\sin\phi_3\cos\phi_3}$	$-\frac{kD_{,\theta}}{RD}$	0
$b_{15}=$	$-k\frac{D_{,\theta}}{D}\cot\phi$	$-\frac{k\tan\phi_3}{x^2D}D_{,\theta}$	0	0
$b_{16}=$	$(1+\nu)-2k(1-\nu)\left[1-\frac{\cot\phi}{D(1-\nu)}(D(1-\nu))_{,\phi}\right]$	$\frac{1}{x\tan\phi_3}-\frac{2k\tan\phi_3}{x^3}(1-\nu)\left[1-\frac{x(D(1-\nu))_{,x}}{D(1-\nu)}\right]$	$\frac{1}{R}$	0
$b_{17}=$	$\frac{\bar h^2}{D}\left(\frac{D(1+\nu)}{\bar h^2}\right)_{,\theta}$	$\frac{\bar h^2}{x\tan\phi_3 D}\left(\frac{D}{\bar h^2}\right)_{,\theta}$	$\frac{\bar h^2}{RD}\left(\frac{D}{\bar h^2}\right)_{,\theta}$	0

Table 3-5-2 Coefficients a_i , b_i , and c_i of Specific Shells (Cont.)

	SPHERE	CONE	CYLINDER	PLATE
$c_1=$	$\frac{\sin\phi}{R^2}D$	0	0	0
$c_2=$	$(2-\nu)\frac{D}{R^2\sin\phi}$	0	0	0
$c_3=$	$2\frac{D}{R^2}\left[\cos\phi+\sin\phi\frac{D_{,\phi}}{D}\right]$	0	0	0
$c_4=$	$2\frac{D_{,\theta}}{R^2\sin\phi}$	0	0	0
$c_5=$	$\frac{D}{R^2\sin^2\phi}\left[\cos\phi+2\frac{\sin\phi}{D}(D(1-\nu))_{,\phi}\right]$	0	0	0
$c_6=$	$\frac{D}{R^2}\left[-\frac{1+\nu\sin^2\phi}{\sin\phi}+\frac{\cos\phi}{D}\left(D(2+\nu)\right)_{,\phi}+\sin\phi\frac{D_{,\phi\phi}}{D}+\frac{(\nu D)_{,\theta\theta}}{D\sin\phi}-\frac{\sin\phi}{k}(1+\nu)\right]$	$-\frac{\nu D\cos^2\phi_3}{k\sin^2\phi_3}$	$-\frac{\nu D}{kR^2}$	0
$c_7=$	$\frac{2}{R^2\sin\phi}\left[\cot\phi D_{,\theta}+(D(1-\nu))_{,\phi\theta}\right]$	0	0	0
$c_8=$	$\cos\phi\frac{D}{R^2}\left[1-\nu-\frac{1+\nu}{k}+\frac{1}{\sin^2\phi}-\frac{1}{D}\left\{2\frac{\sin\phi(\nu D)_{,\phi}}{\cos\phi}+\frac{D_{,\phi}}{\tan\phi}-(\nu D)_{,\phi\phi}-\frac{D_{,\theta\theta}}{\sin\phi}\right\}\right]$	$-\frac{D\cos^3\phi_3}{xk\sin^2\phi_3}$	0	0
$c_9=$	$(2-\nu)\frac{D}{R^2}$	$(2-\nu)\frac{D}{x\tan\phi_3}$	$(2-\nu)\frac{D}{R}$	0
$c_{10}=$	$\frac{D}{R^2\sin^2\phi}$	$\frac{D}{x^3\tan\phi_3\sin^2\phi_3}$	$\frac{D}{R^3}$	0
$c_{11}=$	$\frac{2}{R^2}(D(1-\nu))_{,\theta}$	$\frac{2}{x\tan\phi_3}(D(1-\nu))_{,\theta}$	$\frac{2}{R}(D(1-\nu))_{,\theta}$	0
$c_{12}=$	$-\frac{D}{R^2}\left[\cot\phi-2\frac{D_{,\phi}}{D}\right]$	$-\frac{D}{x^2\tan\phi_3}\left[3-2x\frac{D_{,x}}{D}\right]$	$2\frac{D_{,x}}{R}$	0
$c_{13}=$	$2\frac{D_{,\theta}}{R^2\sin^2\phi}$	$\frac{2D_{,\theta}}{x^3\sin^2\phi_3\tan\phi_3}$	$2\frac{D_{,\theta}}{R^3}$	0
$c_{14}=$	$\frac{2}{R^2}(D(1-\nu))_{,\phi\theta}$	$-\frac{2}{x^2\tan\phi_3}\left[(D(1-\nu))_{,\theta}-x(D(1-\nu))_{,x\theta}\right]$	$\frac{2}{R}(D(1-\nu))_{,x\theta}$	0
$c_{15}=$	$\frac{D}{R^2}\left[2(1-\nu)-\frac{1+\nu}{k}+\frac{1}{\sin^2\phi}-\frac{\cos\phi}{D\sin\phi}(D(3-2\nu))_{,\phi}+R\frac{(\nu D)_{,\phi\phi}}{D}+\frac{D_{,\theta\theta}}{D\sin^2\phi}\right]$	$-\frac{D}{kx\tan\phi_3}+\frac{D}{x^2\tan\phi_3}\left[4-3x\frac{D_{,x}}{D}+x^2\frac{(\nu D)_{,xx}}{D}+\frac{D_{,\theta\theta}}{\sin^2\phi_3 D}\right]$	$\frac{1}{R^3}\left[D_{,\theta\theta}+R^2(\nu D)_{,xx}\right]-\frac{D}{kR^3}$	0
$c_{16}=$	$\frac{2}{R^2}\left[(D(1-\nu))_{,\theta}-\cot\phi(D(1-\nu))_{,\phi\theta}\right]$	$\frac{2}{x^3\tan\phi_3}\left[(D(1-\nu))_{,\theta}-x(D(1-\nu))_{,x\theta}\right]$	0	0

Table 3-5-2 Coefficients a_i , b_i , and c_i of Specific Shells (Cont.)

	SPHERE	CONE	CYLINDER	PLATE
$C_{17}=$	$-\frac{D}{R^2}\sin\phi$	$-X\sin\phi_3 D$	$-RD$	$-D$
$C_{18}=$	$-2\frac{D}{R^2\sin\phi}$	$-2\frac{D}{X\sin\phi_3}$	$-2\frac{D}{R}$	$-2D$
$C_{19}=$	$-\frac{D}{R^2\sin^3\phi}$	$-\frac{D}{X^3\sin^3\phi_3}$	$-\frac{D}{R^3}$	$-D$
$C_{20}=$	$-2\frac{D}{R^2}\left[\cos\phi+\frac{\sin\phi}{D}D_{,\phi}\right]$	$-2\sin\phi_3 D\left[1+x\frac{D_{,x}}{D}\right]$	$-2RD_{,x}$	$-2D_{,x}$
$C_{21}=$	$-2\frac{D_{,\theta}}{R^2\sin\phi}$	$-2\frac{D_{,\theta}}{X\sin\phi_3}$	$-\frac{2}{R}D_{,\theta}$	$-2D_{,y}$
$C_{22}=$	$2\frac{D}{R^2\sin\phi}\left[\cot\phi-\frac{D_{,\phi}}{D}\right]$	$\frac{2D}{X^2\sin\phi_3}\left[1-x\frac{D_{,x}}{D}\right]$	$-\frac{2}{R}D_{,x}$	$-2D_{,x}$
$C_{23}=$	$-\frac{2D_{,\theta}}{R^2\sin^3\phi}$	$-2\frac{D_{,\theta}}{X^3\sin^3\phi_3}$	$-2\frac{D_{,\theta}}{R^3}$	$-2D_{,y}$
$C_{24}=$	$\frac{D}{R^2}\left[\frac{1+\nu\sin^2\phi}{\sin\phi}-(2+\nu)\cos\phi\frac{D_{,\phi}}{D}-\sin\phi\frac{D_{,\phi\phi}}{D}-\frac{(\nu D)_{,\theta\theta}}{\sin\phi D}\right]$	$\sin\phi_3\frac{D}{X}\left[1-x\frac{(D(2+\nu))_{,x}}{D}-x^2\frac{D_{,xx}}{D}-\frac{(\nu D)_{,\theta\theta}}{D\sin^2\phi_3}\right]$	$-\frac{1}{R}\left[(\nu D)_{,\theta\theta}+R^2D_{,xx}\right]$	$-(\nu D)_{,yy}-D_{,xx}$
$C_{25}=$	$-\frac{2}{R^2\sin\phi}\left[\cot\phi(\nu D)_{,\theta}+(D(1-\nu))_{,\phi\theta}\right]$	$-\frac{2}{X^2\sin\phi_3}\left[(\nu D)_{,\theta}+x(D(1-\nu))_{,x\theta}\right]$	$-\frac{2}{R}(D(1-\nu))_{,x\theta}$	$-2(D(1-\nu))_{,xy}$
$C_{26}=$	$-\frac{D}{R^2}\left[\frac{4-(1+\nu)\sin^2\phi}{\sin^3\phi}-3\frac{\cos\phi D_{,\phi}}{\sin^2\phi D}-\frac{(\nu D)_{,\phi\phi}}{\sin\phi D}-\frac{D_{,\theta\theta}}{D\sin^3\phi}\right]$	$-\frac{D}{X^3\sin\phi_3}\left[4-3x\frac{D_{,x}}{D}+x^2\frac{(\nu D)_{,xx}}{D}+\frac{D_{,\theta\theta}}{D\sin\phi_3}\right]$	$-\frac{1}{R}\left[(\nu D)_{,xx}+\frac{D_{,\theta\theta}}{R^2}\right]$	$-(\nu D)_{,xx}-D_{,yy}$
$C_{27}=$	$-\cos\phi\frac{D}{R^2}\left[1-\nu+\frac{1}{\sin^2\phi}+\frac{1}{D}\left\{\frac{\cos\phi D_{,\phi}+2\frac{\sin^2\phi}{\cos\phi}(\nu D)_{,\phi}}{\sin\phi}-(\nu D)_{,\phi\phi}-\frac{D_{,\theta\theta}}{\sin^2\phi}\right\}\right]$	$-\sin\phi_3\frac{D}{2X^2}\left[1-x\frac{D_{,x}}{D}+x^2\frac{(\nu D)_{,xx}}{D}+\frac{D_{,\theta\theta}}{D\sin^2\phi_3}\right]$	0	0
$C_{28}=$	$\frac{2}{R^2}\left[\frac{\cos\phi}{\sin^2\phi}(D(1-\nu))_{,\phi\theta}-\frac{(D(1-\nu))_{,\theta}}{\sin^3\phi}\right]$	$-\frac{2}{X^3\sin\phi_3}\left[(D(1-\nu))_{,\theta}-x(D(1-\nu))_{,x\theta}\right]$	0	0
$C_{29}=$	$-2\frac{D}{kR^2}\sin\phi(1+\nu)$	$-\frac{D\sin\phi_3}{xk\tan^4\phi_3}$	$-\frac{D}{kR^3}$	0
k	$\frac{h^2}{12R^2}$	$\frac{h^2}{12\tan^2\phi_3}$	$\frac{h^2}{12R^2}$	

some of the expressions listed in Table 3-5-2 can be further simplified because derivatives involving the material constants and shell thickness are either identically equal to zero or can be differentiated, and consequently one can express equations (3-4-18) explicitly. The governing differential equations of five specific shell elements so obtained, namely: tapered cone, uniform cone, uniform sphere, uniform cylinder and uniform plate are given by equations (3-5-1) through (3-5-5). By "tapered" we mean the thickness varies linearly along the generators whereas "uniform" means the thickness is constant.

Case 1 Tapered Cone:

$$x\sin\phi_3 u_{,xx}+\frac{1-\nu}{2x\sin\phi_3}u_{,\theta\theta}+\sin\phi_3\left(1+h_1\frac{x}{h}\right)u_{,x}-\frac{\sin\phi_3}{x}\left(1-\nu h_1\frac{x}{h}\right)u+\frac{1+\nu}{2}v_{,x\theta}-\left(\frac{3-\nu}{2x}-h_1\frac{\nu}{h}\right)v_{,\theta}+\nu\cos\phi_3 w_{,x}-\frac{\cos\phi_3}{x}\left(1-\nu h_1\frac{x}{h}\right)w$$

$$=-\frac{(1-\nu^2)}{E\,h}x\sin\phi_3\left(P_x+N^T_{,x}\right)$$

(3-5-1a)

$$\frac{1+\nu}{2}u_{,x\theta}+\frac{1}{2x}\left[3-\nu+h_1(1-\nu)\frac{x}{h}\right]u_{,\theta}+\frac{1-\nu}{2}x\sin\phi_3\left[1+\frac{h^2}{3x^2\tan^2\phi_3}\right]v_{,xx}+\frac{1}{x\sin\phi_3}\left[1+\frac{h^2}{12x^2\tan^2\phi_3}\right]v_{,\theta\theta}$$

$$+\frac{1-\nu}{2}\sin\phi_3\left[1+h_1\frac{x}{h}-\frac{h^2}{3x^2\tan^2\phi_3}+\frac{h_1}{\tan^2\phi_3}\frac{h}{x}\right]v_{,x}-\frac{1-\nu}{2}\frac{\sin\phi_3}{x}\left[1+h_1\frac{x}{h}-\frac{h^2}{3x^2\tan^2\phi_3}+\frac{h_1}{\tan^2\phi_3}\frac{h}{x}\right]v-(2-\nu)\frac{h^2}{12x\tan\phi_3}w_{,xx\theta}$$

$$-\frac{h^2}{12x^3\sin^2\phi_3\tan\phi_3}w_{,\theta\theta\theta}+\frac{h^2}{12x^2\tan\phi_3}\left[1-2\nu-6(1-\nu)h_1\frac{x}{h}\right]w_{,x\theta}+\left(\frac{1}{x\tan\phi_3}-\frac{1-\nu}{6}\frac{h^2}{x^3\tan\phi_3}\left[1-3h_1\frac{x}{h}\right]\right)w_{,\theta}$$

$$=-\frac{1-\nu^2}{E\,h}\left(x\sin\phi_3 P_\theta+N^T_{,\theta}+\frac{M^T_{,\theta}}{x\tan\phi_3}\right)$$

(3-5-1b)

$$-\nu\cos\phi_3 u_{,x}-\frac{\cos\phi_3}{x}u+(2-\nu)\frac{h^2}{12x\tan\phi_3}v_{,xx\theta}+\frac{h^2\cos\phi_3}{12x^3\sin^3\phi_3}v_{,\theta\theta\theta}-\frac{h^2}{4x^2\tan\phi_3}\left(1-2h_1\frac{x}{h}\right)v_{,x\theta}$$

$$-\left(\frac{1}{x\tan\phi_3}+\frac{h^2}{12x^3\tan\phi_3}\left[4-9h_1\frac{x}{h}+6\nu h_1^2\frac{x^2}{h^2}\right]\right)v_{,\theta}-x\sin\phi_3\frac{h^2}{12}w_{,xxxx}-\frac{h^2}{6x\sin\phi_3}w_{,xx\theta\theta}-\frac{h^2}{12x^3\sin^3\phi_3}w_{,\theta\theta\theta\theta}-\sin\phi_3\frac{h^2}{6}\left[1+3h_1\frac{x}{h}\right]w_{,xxx}$$

$$+\frac{h^2}{6x^2\sin\phi_3}\left[1-3h_1\frac{x}{h}\right]w_{,x\theta\theta}+\frac{h^2}{12x}\sin\phi_3\left[1-3(2+\nu)h_1\frac{x}{h}-6h_1^2\frac{x^2}{h^2}\right]w_{,xx}-\frac{h^2}{12x^3\sin\phi_3}\left[4-9h_1\frac{x}{h}+6\nu h_1^2\frac{x^2}{h^2}\right]w_{,\theta\theta}$$

$$-\sin\phi_3\frac{h^2}{12x^2}\left[1-3h_1\frac{x}{h}+6\nu h_1^2\frac{x^2}{h^2}\right]w_{,x}-\frac{\cos^2\phi_3}{x\sin\phi_3}w=-x\sin\phi_3\frac{1-\nu^2}{E\,h}\left(P_z-\frac{N^T}{x\tan\phi_3}+M^T_{,xx}+\frac{M^T_{,x}}{x}+\frac{M^T_{,\theta\theta}}{x^2\sin^2\phi_3}\right)$$

(3-5-1c)

where $h=h_0+h_1x$

Case 2 Uniform Cone:

$$x\sin\phi_3\, u_{,xx} + \frac{1-\nu}{2x\sin\phi_3} u_{,\theta\theta} + \sin\phi_3\, u_{,x} - \frac{\sin\phi_3}{x} u + \frac{1+\nu}{2} v_{,x\theta} - \frac{3-\nu}{2x} v_{,\theta} + \nu\cos\phi_3\, w_{,x} - \frac{\cos\phi_3}{x} w =$$
$$-\frac{h^2 x\sin\phi_3}{12D}\left(P_x + N^T_{,x}\right) \qquad (3\text{-}5\text{-}2a)$$

$$\frac{1+\nu}{2} u_{,x\theta} + \frac{3-\nu}{2x} u_{,\theta} + x\sin\phi_3 \frac{1-\nu}{2}\left(1+\frac{4k}{x^2}\right) v_{,xx} + \frac{1}{x\sin\phi_3}\left(1+\frac{k}{x^2}\right) v_{,\theta\theta} + \sin\phi_3 \frac{1-\nu}{2}\left(1-\frac{4k}{x^2}\right) v_{,x} - \sin\phi_3 \frac{1-\nu}{2x}\left(1-\frac{4k}{x^2}\right) v$$
$$-(2-\nu)k\frac{\tan\phi_3}{x} w_{,xx\theta} - \frac{k}{x^3\sin\phi_3\cos\phi_3} w_{,\theta\theta\theta} + \frac{k\tan\phi_3}{x^2}(1-2\nu) w_{,x\theta} + \left[\frac{1}{x\tan\phi_3} - 2\frac{k\tan\phi_3}{x^3}(1-\nu)\right] w_{,\theta}$$
$$= -\frac{h^2}{12D}\left(x\sin\phi_3\, P_\theta + N^T_{,\theta} + \frac{M^T_{,\theta}}{x\tan\phi_3}\right) \qquad (3\text{-}5\text{-}2b)$$

$$-\frac{\nu D\cos^3\phi_3}{k\sin^2\phi_3} u_{,x} - \frac{D\cos^3\phi_3}{x k\sin^2\phi_3} u + (2-\nu)\frac{D}{x\tan\phi_3} v_{,xx\theta} + \frac{D}{x^3\tan\phi_3\sin^2\phi_3} v_{,\theta\theta\theta} - \frac{3D}{x^2\tan\phi_3} v_{,x\theta}$$
$$-\left[\frac{D}{k x\tan^3\phi_3} - \frac{4D}{x^3\tan\phi_3}\right] v_{,\theta} - x\sin\phi_3\, D w_{,xxxx} - 2\frac{D}{x\sin\phi_3} w_{,xx\theta\theta} - \frac{D}{x^3\sin^3\phi_3} w_{,\theta\theta\theta\theta} - 2\sin\phi_3\, D w_{,xxx}$$
$$+\frac{2D}{x^2\sin\phi_3} w_{,x\theta\theta} + \sin\phi_3 \frac{D}{x} w_{,xx} - \frac{4D}{x^3\sin\phi_3} w_{,\theta\theta} - \sin\phi_3\frac{D}{x^2} w_{,x} - \frac{D\sin\phi_3}{x k\tan^4\phi_3} w =$$
$$-x\sin\phi_3\left(P_z - \frac{N^T}{x\tan\phi_3} + M^T_{,xx} + \frac{M^T_{,x}}{x} + \frac{M^T_{,\theta\theta}}{x^2\sin^2\phi_3}\right) \qquad (3\text{-}5\text{-}2c)$$

where $k = \dfrac{h^2}{12\tan^2\phi_3}$

Case 3 Uniform Sphere:

$$\sin\phi(1+k)u_{,\phi\phi}+\frac{1-\nu}{2\sin\phi}(1+4k)u_{,\theta\theta}+\cos\phi(1+k)u_{,\phi}-\frac{\cos^2\phi+\nu\sin^2\phi}{\sin\phi}(1+k)u+\left[\frac{1+\nu}{2}+(2-\nu)k\right]v_{,\phi\theta}$$

$$-\frac{\cot\phi}{2}\left[3-\nu+2k(3-2\nu)\right]v_{,\theta}-k\sin\phi\, w_{,\phi\phi\phi}-(2-\nu)\frac{k}{\sin\phi}w_{,\phi\theta\theta}-k\cos\phi\, w_{,\phi\phi}+\frac{k}{\sin\phi}(3-\nu)\cot\phi\, w_{,\theta\theta}$$

$$+\left[(1+\nu)\sin\phi+\frac{\cos^2\phi+\nu\sin^2\phi}{\sin\phi}k\right]w_{,\phi}=-\frac{h^2}{12D}R\sin\phi\left(RP_\phi+N^T_{,\phi}+\frac{M^T_{,\phi}}{R}\right)$$

(3-5-3a)

$$\left[\frac{1+\nu}{2}+(2-\nu)k\right]u_{,\phi\theta}+\cot\phi\left[\frac{3-\nu}{2}+(3-2\nu)k\right]u_{,\theta}+\sin\phi\frac{1-\nu}{2}(1+4k)v_{,\phi\phi}+\frac{1+k}{\sin\phi}v_{,\theta\theta}+\cos\phi\frac{(1-\nu)}{2}(1+4k)v_{,\phi}$$

$$+\frac{1-\nu}{2}\left(\frac{\sin^2\phi-\cos^2\phi}{\sin\phi}\right)(1+4k)v-(2-\nu)k\,w_{,\phi\phi\theta}-\frac{k}{\sin^2\phi}w_{,\theta\theta\theta}-k\cot\phi\, w_{,\phi\theta}+\left[(1+\nu)-2k(1-\nu)\right]w_{,\theta}$$

$$=-\frac{h^2R}{12D}\left(R\sin\phi\, P_\theta+N^T_{,\theta}+\frac{M^T_{,\theta}}{R}\right)$$

(3-5-3b)

$$\frac{\sin\phi}{R^2}Du_{,\phi\phi\phi}+(2-\nu)\frac{D}{R^2\sin\phi}u_{,\phi\theta\theta}+2\frac{D}{R^2}\cos\phi\, u_{,\phi\phi}+\frac{D\cos\phi}{R^2\sin^2\phi}u_{,\theta\theta}-\frac{D}{R^2}\left[\frac{1+\nu\sin^2\phi}{\sin\phi}+\frac{\sin\phi}{k}(1+\nu)\right]u_{,\phi}$$

$$+\cos\phi\frac{D}{R^2}\left[1-\nu-\frac{1+\nu}{k}+\frac{1}{\sin^2\phi}\right]u+(2-\nu)\frac{D}{R^2}v_{,\phi\phi\theta}+\frac{D}{R^2\sin^2\phi}v_{,\theta\theta\theta}-\frac{D}{R^2}\cot\phi\, v_{,\phi\theta}+\frac{D}{R^2}\left[2(1-\nu)-\frac{1+\nu}{k}+\frac{1}{\sin^2\phi}\right]v_{,\theta}$$

$$-\frac{D}{R^2}\sin\phi\, w_{,\phi\phi\phi\phi}-2\frac{D}{R^2\sin\phi}w_{,\phi\phi\theta\theta}-\frac{D}{R^2\sin^3\phi}w_{,\theta\theta\theta\theta}-2\frac{D}{R^2}\cos\phi\, w_{,\phi\phi\phi}+2\frac{D\cos\phi}{R^2\sin^2\phi}w_{,\phi\theta\theta}+\frac{D}{R^2}\left[\frac{1+\nu\sin^2\phi}{\sin\phi}\right]w_{,\phi\phi}$$

$$-\frac{D}{R^2}\left[\frac{4-(1+\nu)\sin^2\phi}{\sin^3\phi}\right]w_{,\theta\theta}-\cos\phi\frac{D}{R^2}\left(1-\nu+\frac{1}{\sin^2\phi}\right)w_{,\phi}-2\frac{D}{kR^2}\sin\phi(1+\nu)w=-R^2\sin\phi\left(P_z-2\frac{N^T}{R}+\frac{M^T_{,\phi\phi}}{R^2}+\frac{\cos\phi}{R^2\sin\phi}M^T_{,\phi}+\frac{M^T_{,\theta\theta}}{R^2\sin^2\phi}\right)$$

where $k=\dfrac{h^2}{12R^2}$ (3-5-3c)

Case 4 Uniform Cylinder:

$$R u_{,xx} + \frac{1-\nu}{2R} u_{,\theta\theta} + \frac{1+\nu}{2} v_{,x\theta} + \nu w_{,x} = -\frac{R h^2}{12D}\left(P_x + N^T_{,x}\right) \qquad (3\text{-}5\text{-}4a)$$

$$\frac{1+\nu}{2} u_{,x\theta} + \frac{1-\nu}{2} R\,(1+4k) v_{,xx} + \frac{(1+k)}{R} v_{,\theta\theta} - (2-\nu) k R\, w_{,xx\theta} - \frac{k}{R} w_{,\theta\theta\theta} + \frac{w_{,\theta}}{R} = -\frac{h^2}{12D}\left(R P_\theta + N^T_{,\theta} + \frac{M^T_{,\theta}}{R}\right) \qquad (3\text{-}5\text{-}4b)$$

$$-\frac{\nu D}{kR^2} u_{,x} + (2-\nu)\frac{D}{R} v_{,xx\theta} + \frac{D}{R^3} v_{,\theta\theta\theta} - \frac{D}{kR^3} v_{,\theta} - RD\, w_{,xxxx} - 2\frac{D}{R} w_{,xx\theta\theta} - \frac{D}{R^3} w_{,\theta\theta\theta\theta} - \frac{D}{kR^3} w =$$

$$-R\left(P_z - \frac{N^T}{R} + M^T_{,xx} + \frac{M^T_{,\theta\theta}}{R^2}\right) \qquad (3\text{-}5\text{-}4c)$$

where $k = \dfrac{h^2}{12R^2}$

Case 5 Uniform Plate:

$$u_{,xx} + \frac{1-\nu}{2} u_{,yy} + \frac{1+\nu}{2} v_{,xy} = -\frac{h^2}{12D}\left(P_x + N^T_{,x}\right) \qquad (3\text{-}5\text{-}5a)$$

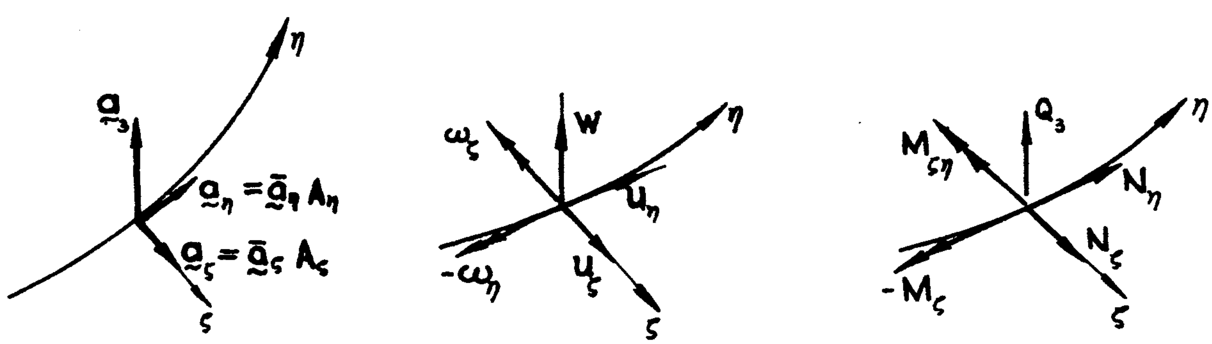
$$\frac{1+\nu}{2} u_{,xy} + \frac{1-\nu}{2} v_{,xx} + v_{,yy} = -\frac{h^2}{12D}\left(P_y + N^T_{,y}\right) \qquad (3\text{-}5\text{-}5b)$$

$$D w_{,xxxx} + 2D w_{,xxyy} + D w_{,yyyy} = \left(P_z + M^T_{,xx} + M^T_{,yy}\right) \qquad (3\text{-}5\text{-}5c)$$

3.6 Boundary Conditions

A unique solution of the equilibrium equations (3-3-26), constitutive equations (3-3-30), and the strain-displacement equations (3-3-12); (3-3-14) for stresses and displacements of a shell is determined by the boundary conditions. These boundary conditions are given by a certain number of relations between forces, moments, displacements or functions of these quantities at the edge of the shell.

We let the boundary of the shell be a smooth curve c and introduce a system of orthogonal curvilinear coordinates (η , ζ) , in which the curve c is given by the equation ζ = constant . Along the boundary curve



a. Coordinates b. Deformations c. Forces

Fig. 3-4 Boundary Coordinates, Deformations and Forces

the displacement and rotation components are shown in Fig. 3-4b and the stress resultants and couples are shown in Fig. 3-4c.

Then, the force, moment, displacement and rotation vectors at the boundary are

$$\underset{\sim}{N} = \bar{\underset{\sim}{a}}_\eta N_\eta + \bar{\underset{\sim}{a}}_\zeta N_\zeta + \underset{\sim}{a}_3 Q_3 \quad ,$$

$$\underset{\sim}{M} = \bar{\underset{\sim}{a}}_\eta M_\zeta - \bar{\underset{\sim}{a}}_\zeta M_{\zeta\eta} \quad ,$$

$$\underset{\sim}{u} = \bar{\underset{\sim}{a}}_\eta u_\eta + \bar{\underset{\sim}{a}}_\zeta u_\zeta + \underset{\sim}{a}_3 w \quad , \qquad (3\text{-}6\text{-}1\ a\text{-}d)$$

$$\underset{\sim}{\Omega} = \bar{\underset{\sim}{a}}_\eta \omega_\eta - \bar{\underset{\sim}{a}}_\zeta \omega_\zeta \quad .$$

The force and moment vectors are a function of five quantities N_η , N_ζ , Q_3 , M_ζ , $M_{\zeta\eta}$. Thus the number of boundary conditions should be five. However, because of the Kirchhoff-Love hypothesis, the number of boundary conditions is actually four. To find the correct boundary conditions we consider the work done by the boundary forces and displacements

$$W^b = \int_c \left[\underset{\sim}{u}\cdot\underset{\sim}{N} + \underset{\sim}{\Omega}\cdot\underset{\sim}{M}\right] ds \ . \qquad (3\text{-}6\text{-}2)$$

Expanding the scalar products in the above equation by use of equations (3-6-1), we obtain

$$W^b = \int \left[N_\eta u_\eta + N_\zeta u_\zeta + Q_3 w + M_\zeta \omega_\eta + M_{\zeta\eta}\omega_\zeta\right] ds \ . \qquad (3\text{-}6\text{-}3)$$

The rotation ω_ζ can be written in terms of the displacement components by

$$\omega_\zeta = -\frac{1}{A_\eta}\frac{\partial w}{\partial \eta} + \frac{u_\eta}{R} - \frac{u_\zeta}{R_{12}} \quad ,$$

where

$$\frac{1}{R} = \left(\frac{\cos^2\lambda}{r_1} + \frac{\sin^2\lambda}{r_2}\right) \quad ,$$

$$\frac{1}{R_{12}} = \frac{\sin 2\lambda}{2}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \quad ,$$

λ - angle between $\bar{\underset{\sim}{a}}_\eta$ and $\bar{\underset{\sim}{a}}_1$,

A_η - defined by $ds = A_\eta d\eta$.

With the relation (3-6-4), equation (3-6-3) becomes

$$W^b = \int_c \left[\left(N_\eta + \frac{M_{\zeta\eta}}{R}\right) u_\eta + \left(N_\zeta - \frac{M_{\zeta\eta}}{R_{12}}\right) u_\zeta + Q_3 w + M_\zeta \omega_\eta - \frac{M_{\zeta\eta}}{A_\eta} \frac{\partial w}{\partial \eta} \right] ds \,. \quad (3\text{-}6\text{-}5)$$

The last term in the integrand of this equation can be integrated by parts to yield

$$-\int_c M_{\zeta\eta} \frac{\partial w}{\partial \eta} d\eta = -M_{\zeta\eta} w \Big|_{\eta_1}^{\eta_2} + \int_c \frac{\partial M_{\zeta\eta}}{\partial \eta} w \, d\eta \quad .$$

If $M_{\zeta\eta} W$ is single-valued, equation (3-6-5) becomes

$$W^b = \int_c \left[\left(N_\eta + \frac{M_{\zeta\eta}}{R}\right) u_\eta + \left(N_\zeta - \frac{M_{\zeta\eta}}{R_{12}}\right) u_\zeta + \left(Q_3 + \frac{1}{A_\eta} \frac{\partial M_{\zeta\eta}}{\partial \eta}\right) w + M_\zeta \omega_\eta \right] ds \,. \quad (3\text{-}6\text{-}6)$$

The proper four boundary conditions are given by equation (3-6-6). Thus, the required boundary forces and moments are

$$\begin{aligned}
\bar{N}_\eta &= N_\eta + \frac{M_{\zeta\eta}}{R} \,, \\
\bar{N}_\zeta &= N_\zeta - \frac{M_{\zeta\eta}}{R_{12}} \,, \\
\bar{Q} &= Q_3 + \frac{1}{A_\eta} \frac{\partial M_{\zeta\eta}}{\partial \eta} \,, \\
\bar{M}_\zeta &= M_\zeta \,.
\end{aligned} \qquad (3\text{-}6\text{-}7a\text{-}d)$$

or if the boundary conditions are in terms of displacements, the required conditions are

$$\bar{u}_\eta = u_\eta \quad ,$$

$$\bar{u}_\zeta = u_\zeta \quad , \qquad (3\text{-}6\text{-}8\ \text{a-d})$$

$$\bar{w} = w \quad ,$$

$$\bar{\omega}_\eta = \omega_\eta \quad .$$

At a boundary the force, moment, displacement, and rotation vectors are

$$\bar{\underset{\sim}{N}} = \bar{\underset{\sim}{a}}_\eta \bar{N}_\eta + \bar{\underset{\sim}{a}}_\zeta \bar{N}_\zeta + \underset{\sim}{a}_3 \bar{Q} \quad ,$$

$$\bar{\underset{\sim}{M}} = \bar{\underset{\sim}{a}}_\eta \bar{M}_\zeta \quad ,$$

$$\bar{\underset{\sim}{u}} = \bar{\underset{\sim}{a}}_\eta \bar{u}_\eta + \bar{\underset{\sim}{a}}_\zeta \bar{u}_\zeta + \underset{\sim}{a}_3 \bar{w} \quad , \qquad (3\text{-}6\text{-}9\ \text{a-d})$$

$$\bar{\underset{\sim}{\Omega}} = \bar{\underset{\sim}{a}}_\eta \bar{\omega}_\eta \quad .$$

Certain specific boundary conditions are

a) Fixed edge

A fixed edge is defined by

$$\bar{\underset{\sim}{u}} = \bar{\underset{\sim}{\Omega}} = 0 \quad . \qquad (3\text{-}6\text{-}10\ \text{a,b})$$

In terms of components this becomes

$$u_\eta = u_\zeta = w = \omega_\eta = 0 \quad . \qquad (3\text{-}6\text{-}11\ \text{a-d})$$

However, ω_η can be written in terms of displacement components, so that the conditions for a fixed edge are

$$u_\eta = u_\zeta = w = \frac{\partial w}{\partial \zeta} = 0 \qquad (3\text{-}6\text{-}12a\text{-}d)$$

b) Free edge

At a free edge we have

$$\bar{\underset{\sim}{N}} = \bar{\underset{\sim}{M}} = 0 . \qquad (3\text{-}6\text{-}13a,b)$$

In terms of stress resultants and couples we have

$$\bar{N}_\eta = \bar{N}_\zeta = \bar{Q} = \bar{M}_\zeta = 0 . \qquad (3\text{-}6\text{-}14a\text{-}d)$$

In the case that two shells are joined together and the rigidity at the edge is of the same order of magnitude, the boundary conditions become continuity requirements and are

$$\begin{aligned} \bar{\underset{\sim}{N}}_1 &= \bar{\underset{\sim}{N}}_2 , & \bar{\underset{\sim}{u}}_1 &= \bar{\underset{\sim}{u}}_2 , \\ \bar{\underset{\sim}{M}}_1 &= \bar{\underset{\sim}{M}}_2 , & \bar{\underset{\sim}{\Omega}}_1 &= \bar{\underset{\sim}{\Omega}}_2 , \end{aligned} \qquad (3\text{-}6\text{-}15a\text{-}d)$$

where 1 , 2 refers to shell one and two.

4. MEMBRANE THEORY OF SHELLS

4.1 Introduction

Membrane theory is an approximate method of analysis of thin shells based upon the assumption that all moments are negligible. This assumption is justifiable either when the shell has very small bending resistance or when the changes of curvature and torsion of the middle surface $(\chi_1 , \chi_2 , \chi_{12})$ are very small. The first is found in flexible shells such as a diaphragm while the second case is found in shells having finite bending stiffness but a momentless state of stress. The equations describing the membrane behavior in these two shells are identical. Shells have an advantage over a plate in that transverse loads are sustained without appreciable bending provided its edges are suitably supported, the loads do not vary greatly, and the radius of curvature is smooth. It will be shown that with the assumption of neglecting moments the transverse shear resultants are zero, and the unknown stress resultants are reduced to N_1 , N_2 , N_{12} , N_{21} . Thus, the membrane theory becomes a process of determining these four unknowns and the displacement components (u , v , w) in the absence of moments.

4.2 Shell Equations

The equations of membrane theory can be obtained from the equations of the general theory. When the moments are set equal to zero in equations (3-4-15) it is seen that the transverse shear stress resultants (Q_1 , Q_2) must be zero. Hence, the equilibrium equations for membrane theory are

$$(A_2N_1)_{,1} + (A_1N_{21})_{,2} + N_{12}\,A_{1,2} - N_2\,A_{2,1} + A_1\,A_2\,p_1 = 0\,,$$

$$(A_2N_{12})_{,1} + (A_1N_2)_{,2} + N_{21}\,A_{2,1} - N_1\,A_{1,2} + A_1\,A_2\,p_2 = 0\,,$$

$$\frac{N_1}{r_1} + \frac{N_2}{r_2} - p_3 = 0\,, \qquad (4\text{-}2\text{-}1a\text{-}d)$$

$$N_{12} - N_{21} = 0\,.$$

These equations involve four unknowns. From the last of these equations it is seen that the shear stress resultants N_{12} and N_{21} are equal. This in effect reduces the problem to three unknowns with three equations. Thus, the problem of determining the stress resultants is statically determinate with respect to the equilibrium of an infinitely small element, however, not necessarily with respect to the entire shell.

Since membrane theory is concerned with determining N_1 , N_2 , N_{12} , u , v , w , equations (3-4-16) and strain-displacement relations (3-4-12) respectively are

$$N_1 = \frac{Eh}{1-\nu^2}\left(\bar{\epsilon}_1 + \nu\bar{\epsilon}_2\right) + N^T\,,$$

$$N_2 = \frac{Eh}{1-\nu^2}\left(\bar{\epsilon}_2 + \nu\bar{\epsilon}_1\right) + N^T\,, \qquad (4\text{-}2\text{-}2a\text{-}c)$$

$$N_{12} = N_{12} = hG\bar{\gamma}\,,$$

and

$$\bar{\epsilon}_1 = \frac{1}{A_1}\,u_{1,1} + \frac{A_{1,2}}{A_1A_2}\,u_2 + \frac{u_3}{r_1}\,,$$

$$\bar{\epsilon}_2 = \frac{1}{A_2} u_{2,2} + \frac{A_{2,1}}{A_1 A_2} u_1 + \frac{u_3}{r_2} \quad , \qquad (4\text{-}2\text{-}3a\text{-}c)$$

$$\bar{\gamma} = \frac{A_2}{A_2}\left(\frac{u_2}{A_2}\right)_{,1} + \frac{A_1}{A_2}\left(\frac{u_1}{A_1}\right)_{,2} \quad .$$

Equations (4-2-1), (4-2-2) and (4-2-3) are the complete set of equations for membrane theory. In many cases it is convenient to solve the equations in two steps. The first step would be to solve equation (4-2-1) for N_1 , N_2 , N_{12} with arbitrary functions of integration. These arbitrary functions are determined by the boundary loading. Once N_1 , N_2 , N_{12} are known the strains can be determined by equation (4-2-2). When these strains are substituted into equation (4-2-3), the displacements are obtained by solving this set of equations in two parts, i.e., the displacements for the particular strains and the displacements of the homogeneous set of equations. This last set yields additional functions of integration to satisfy certain displacement boundary conditions.

The method of analysis as presented is in many problems convenient, however not compulsory. A single system of equations in u , v , w can be obtained by expressing the stress resultants in terms of displacements by use of equations (4-2-2) and (4-2-3). Then these stress resultants are substituted into equation (4-2-1) to yield the desired set of equations. It should be noted that identical results can also be obtained directly from the general governing differential equations (3-4-18) by setting $h^2 = 0$. With this type of reduction it is assumed that the shell has finite rigidity in extension and in-plane shear but no thickness to resist bending. If h^2 is set to zero in equations (3-4-18) then the governing equations for membrane shells are given by equations (4-2-4a-c). The coefficients of these equations are given for a general shell (Table 4-2-1) and then reduced to three specific cases (sphere, cone and cylinder). Equations for a tapered cone and constant thickness cones, spheres, and cylinders are presented by equations (4-2-5), (4-2-6), (4-2-7), and (4-2-8) respectively.

$$a_1 u_{1,11} + a_2 u_{1,22} + a_3 u_{1,1} + a_4 u_{1,2} + a_5 u_1 + a_6 u_{2,12} + a_7 u_{2,1} + a_8 u_{2,2} + a_9 u_2$$
$$+ a_{15} u_3 + a_{17} u_3 = -\frac{(1-\nu^2)}{Eh}\left(A_1 A_2 P_1 + A_2 N^T_{,1}\right)$$

(4-2-4 a,b,c)

$$b_1 u_{1,12} + b_2 u_{1,1} + b_3 u_{1,2} + b_4 u_1 + b_5 u_{2,11} + b_6 u_{2,22} + b_7 u_{2,1} + b_8 u_{2,2} + b_9 u_2$$
$$b_{16} u_{3,2} + b_{17} u_3 = -\frac{(1-\nu^2)}{Eh}\left(A_1 A_2 P_2 + A_1 N^T_{,2}\right)$$

$$c_6 u_{1,1} + c_8 u_1 + c_{16} u_{2,2} + c_{16} u_2 + c_{29} u_3 = -A_1 A_2\left(P_3 - \frac{N^T}{r_1} - \frac{N^T}{r_2}\right)$$

Table 4-2-1 Coefficients a_i , b_i , and c_i of Membrane Shells

	GENERAL FORM	SPHERE	CONE	CYLINDER
$a_1=$	$\frac{A_2}{A_1}$	$\sin\phi$	$x\sin\phi_3$	R
$a_2=$	$\frac{1-\nu}{2}\frac{A_1}{A_2}$	$\frac{1-\nu}{2\sin\phi}$	$\frac{1-\nu}{2x\sin\phi_3}$	$\frac{1-\nu}{2R}$
$a_3=$	$\frac{h^2}{D}\left(\frac{A_2D}{A_1h^2}\right)_{,1}$	$\cos\phi+\frac{\sin\phi}{D}h^2\left(\frac{D}{h}\right)_{,\theta}$	$\sin\phi_3\left[1+x\frac{h^2}{D}\left(\frac{D}{h^2}\right)_{,x}\right]$	$R\frac{h^2}{D}\left(\frac{D}{h^2}\right)_{,x}$
$a_4=$	$\frac{h^2}{D}\left(\frac{A_1}{A_2}\frac{D(1-\nu)}{2h^2}\right)_{,2}$	$\frac{1}{2\sin\phi D}\left[h^2\left(\frac{D(1-\nu)}{h^2}\right)_{,\theta}\right]$	$\frac{h^2}{2x\sin\phi_3D}\left(\frac{D(1-\nu)}{h^2}\right)_{,\theta}$	$\frac{h^2}{2RD}\left(\frac{D(1-\nu)}{h^2}\right)_{,\theta}$
$a_5=$	$A_2\frac{h^2}{D}\left(\frac{A_{2,1}}{A_1A_2}\frac{\nu D}{h^2}\right)_{,1}-\frac{h^2}{A_1D}\left(\frac{A_1A_{1,2}}{2h^2A_2}D(1-\nu)\right)_{,2}-(1-\nu)\frac{A_{2,1}^2}{A_1A_2}$	$\frac{\cos\phi}{D}h^2\left(\frac{\nu D}{h^2}\right)_{,\phi}-\frac{\cos^2\phi+\nu\sin^2\phi}{\sin\phi}$	$-\frac{\sin\phi_3}{x}\left[1-x\frac{h^2}{D}\left(\frac{\nu D}{h^2}\right)_{,x}\right]$	0
$a_6=$	$\frac{1+\nu}{2}$	$\frac{1+\nu}{2}$	$\frac{1+\nu}{2}$	$\frac{1+\nu}{2}$
$a_7=$	$\frac{A_{1,2}}{A_1}+\frac{h^2}{A_1D}\left(\frac{A_1}{2h^2}D(1-\nu)\right)_{,2}$	$\frac{h^2}{2D}\left(\frac{D(1-\nu)}{h^2}\right)_{,\theta}$	$\frac{h^2}{2D}\left(\frac{D(1-\nu)}{h^2}\right)_{,\theta}$	$\frac{h^2}{2D}\left(\frac{D(1-\nu)}{h^2}\right)_{,\theta}$
$a_8=$	$\frac{h^2}{D}\left(\nu\frac{D}{h^2}\right)_{,1}-\frac{3-\nu}{2}\frac{A_{2,1}}{A_2}$	$-\frac{3-\nu}{2}\cot\phi+\frac{h^2}{D}\left(\frac{\nu D}{h^2}\right)_{,\phi}$	$-\frac{3-\nu}{2x}+\frac{h^2}{D}\left(\frac{\nu D}{h^2}\right)_{,x}$	$\frac{h^2}{D}\left(\frac{\nu D}{h^2}\right)_{,x}$
$a_9=$	$A_2\frac{h^2}{2D}\left(\frac{A_{1,2}}{A_1A_2}\frac{D}{h^2}\right)_{,1}-\frac{h^2}{A_1D}\left(\frac{A_1A_{2,1}}{2h^2A_2}D(1-\nu)\right)_{,2}+(1-\nu)\frac{A_{1,2}A_{2,1}}{A_1A_2}$	$-\frac{\cot\phi h^2}{2D}\left(\frac{D(1-\nu)}{h^2}\right)_{,\theta}$	$-\frac{h^2}{2Dx}\left(\frac{D(1-\nu)}{h^2}\right)_{,\theta}$	0
$a_{10}=$	0	0	0	0
$a_{11}=$	0	0	0	0
$a_{12}=$	0	0	0	0
$a_{13}=$	0	0	0	0
$a_{14}=$	0	0	0	0
$a_{15}=$	$A_2\left(\frac{1}{r_1}+\frac{\nu}{r_2}\right)$	$(1+\nu)\sin\phi$	$\nu\cos\phi_3$	ν
$a_{16}=$	0	0	0	0
$a_{17}=$	$\frac{h^2}{A_2D}\left(A_2^2\frac{D}{h^2}\left\{\frac{1}{r_1}+\frac{\nu}{r_2}\right\}\right)_{,1}-(1+\nu)A_{2,1}\left\{\frac{1}{r_1}+\frac{1}{r_2}\right\}$	$\sin\phi\frac{h^2}{D}\left(\frac{D(1+\nu)}{h^2}\right)_{,\phi}$	$-\frac{\cos\phi_3}{x}\left[1-x\frac{h^2}{D}\left(\frac{\nu D}{h^2}\right)_{,x}\right]$	$\frac{h^2}{D}\left(\frac{\nu D}{h^2}\right)_{,x}$

Table 4-2-1 Coefficients a_i , b_i , and c_i of Membrane Shells (Cont.)

	GENERAL FORM	SPHERE	CONE	CYLINDER
$b_1=$	$\frac{1+\nu}{2}$	$\frac{1+\nu}{2}$	$\frac{1+\nu}{2}$	$\frac{1+\nu}{2}$
$b_2=$	$\frac{\bar h^2}{D}\left(\frac{\nu D}{\bar h^2}\right)_{,\xi_2}-\frac{3-\nu}{2}\frac{A_{1,\xi_2}}{A_1}$	$\frac{\bar h^2}{D}\left(\frac{\nu D}{\bar h^2}\right)_{,\theta}$	$\frac{\bar h^2}{D}\left(\frac{\nu D}{\bar h^2}\right)_{,\theta}$	$\frac{\bar h^2}{D}\left(\frac{\nu D}{\bar h^2}\right)_{,\theta}$
$b_3=$	$\frac{\bar h^2}{A_2 D}\left(\frac{A_2}{2\bar h^2}D(1-\nu)\right)_{,\xi_1}+\frac{A_{2,\xi_1}}{A_2}$	$\cot\phi\left(\frac{3-\nu}{2}\right)+\frac{\bar h^2}{2D}\left(\frac{D(1-\nu)}{\bar h^2}\right)_{,\phi}$	$\frac{1}{2x}\left[3-\nu+x\frac{\bar h^2}{D}\left(\frac{D(1-\nu)}{\bar h^2}\right)_{,x}\right]$	$\frac{\bar h^2}{2D}\left(\frac{D(1-\nu)}{\bar h^2}\right)_{,x}$
$b_4=$	$A_1\frac{\bar h^2}{D}\left(\frac{A_{2,\xi_1}}{A_1A_2}\frac{D}{\bar h^2}\right)_{,\xi_2}-\frac{\bar h^2}{A_2D}\left(\frac{A_2A_{1,\xi_2}}{2\bar h^2A_1}D(1-\nu)\right)_{,\xi_1}+(1-\nu)\frac{A_{1,\xi_2}A_{2,\xi_1}}{A_1A_2}$	$\frac{\cot\phi}{D}\bar h^2\left(\frac{D}{\bar h^2}\right)_{,\theta}$	$\frac{\bar h^2}{xD}\left(\frac{D}{\bar h^2}\right)_{,\theta}$	0
$b_5=$	$\frac{1-\nu}{2}\frac{A_2}{A_1}$	$\sin\phi\frac{1-\nu}{2}$	$x\sin\phi_s\frac{1-\nu}{2}$	$\frac{(1-\nu)}{2}R$
$b_6=$	$\frac{A_1}{A_2}$	$\frac{1}{\sin\phi}$	$\frac{1}{x\sin\phi_s}$	$\frac{1}{R}$
$b_7=$	$\frac{\bar h^2}{D}\left(\frac{A_2}{A_1}\frac{D(1-\nu)}{2\bar h^2}\right)_{,\xi_1}$	$\cos\phi\frac{1-\nu}{2}+\frac{\sin\phi}{2D}\bar h^2\left(\frac{D(1-\nu)}{\bar h^2}\right)_{,\phi}$	$\frac{1-\nu}{2}\sin\phi_s\left[1+\frac{\bar h^2x}{D(1-\nu)}\left(\frac{D(1-\nu)}{\bar h^2}\right)_{,x}\right]$	$\frac{R\bar h^2}{2D}\left(\frac{D(1-\nu)}{\bar h^2}\right)_{,x}$
$b_8=$	$\frac{\bar h^2}{D}\left(\frac{A_1D}{A_2\bar h^2}\right)_{,\xi_2}$	$\frac{\bar h^2}{\sin\phi D}\left(\frac{D}{\bar h^2}\right)_{,\theta}$	$\frac{\bar h^2}{x\sin\phi_s D}\left(\frac{D}{\bar h^2}\right)_{,\theta}$	$\frac{\bar h^2}{RD}\left(\frac{D}{\bar h^2}\right)_{,\theta}$
$b_9=$	$A_1\frac{\bar h^2}{D}\left(\frac{A_{1,\xi_2}}{A_1A_2}\nu\frac{D}{\bar h^2}\right)_{,\xi_2}-\frac{\bar h^2}{A_2D}\left(\frac{A_2A_{1,\xi_2}}{2\bar h^2A_1}D(1-\nu)\right)_{,\xi_1}-(1-\nu)\frac{A_{1,\xi_2}^2}{A_1A_2}$	$\frac{1-\nu}{2}\left(\frac{\sin^2\phi-\cos^2\phi}{\sin\phi}\right)-\frac{\cos\phi}{2D}\bar h^2\left(\frac{D(1-\nu)}{\bar h^2}\right)_{,\phi}$	$-\frac{\sin\phi_s(1-\nu)}{2x}\left[1+\frac{\bar h^2x}{D(1-\nu)}\left(\frac{D(1-\nu)}{\bar h^2}\right)_{,x}\right]$	0
$b_{10}=$	0	0	0	0
$b_{11}=$	0	0	0	0
$b_{12}=$	0	0	0	0
$b_{13}=$	0	0	0	0
$b_{14}=$	0	0	0	0
$b_{15}=$	0	0	0	0
$b_{16}=$	$A_1\left(\frac{1}{r_2}+\frac{\nu}{r_1}\right)$	$1+\nu$	$\frac{1}{x\tan\phi_s}$	$\frac{1}{R}$
$b_{17}=$	$\frac{\bar h^2}{A_1D}\left(A_1^2\frac{D}{\bar h^2}\left\{\frac{1}{r_2}+\frac{\nu}{r_1}\right\}\right)_{,\xi_2}-(1+\nu)A_{1,\xi_2}\left\{\frac{1}{r_2}+\frac{1}{r_1}\right\}$	$\frac{\bar h^2}{D}\left(\frac{D(1+\nu)}{\bar h^2}\right)_{,\theta}$	$\frac{\bar h^2}{x\tan\phi_s D}\left(\frac{D}{\bar h^2}\right)_{,\theta}$	$\frac{\bar h^2}{RD}\left(\frac{D}{\bar h^2}\right)_{,\theta}$

Table 4-2-1 Coefficients a_i , b_i , and c_i
of Membrane Shells (Cont.)

	GENERAL FORM	SPHERE	CONE	CYLINDER
$c_6 =$	$-12 \frac{D}{h^2 r_1} A_2 \left(1+\nu \frac{r_1}{r_2}\right)$	$-\frac{D \sin\phi}{k R^2}(1+\nu)$	$-\frac{\nu D \cos^3\phi_3}{k \sin\phi_3}$	$-\frac{\nu D}{k R^2}$
$c_8 =$	$-12 \frac{D}{h^2 r_2} A_{2,1} \left(1+\nu \frac{r_2}{r_1}\right)$	$-\cos\phi \frac{(1+\nu) D}{k R^2}$	$-\frac{D \cos^3\phi_3}{x k \sin^2\phi_3}$	0
$c_{13} =$	$-12 \frac{D}{h^2 r_2} A_1 \left(1+\nu \frac{r_2}{r_1}\right)$	$-\frac{(1+\nu) D}{k R^2}$	$-\frac{D}{k x \tan^3\phi_3}$	$-\frac{D}{k R^3}$
$c_{14} =$	$-12 \frac{D}{h^2 r_1} A_{1,2} \left(1+\nu \frac{r_1}{r_2}\right)$	0	0	0
$c_{20} =$	$-12 A_1 A_2 \frac{D}{h^2} \left(\frac{r_1^2 + 2\nu r_1 r_2 + r_2^2}{r_1^2 r_2^2}\right)$	$-2 \frac{D}{k R^2} \sin\phi (1+\nu)$	$-\frac{D \sin\phi_3}{x k \tan^2\phi_3}$	$-\frac{D}{k R^3}$
$k =$		$\frac{h^2}{12 R^2}$	$\frac{h^2}{12 \tan^2\phi_3}$	$\frac{h^2}{12 R^2}$

Case 1 Tapered Cone:

$$x\sin\phi_s u_{,xx} + \frac{1-\nu}{2x\sin\phi} u_{,\theta\theta} + \sin\phi_s\left(1+h_1\frac{x}{h}\right)u_{,x} - \frac{\sin\phi_s}{x}\left(1-\nu h_1\frac{x}{h}\right)u + \frac{1+\nu}{2}v_{,x\theta} - \left(\frac{3-\nu}{2x} - h_1\frac{x}{h}\right)v_{,\theta}$$

$$+\nu\cos\phi_s w_{,x} - \frac{\cos\phi_s}{x}\left(1-\nu h_1\frac{x}{h}\right)w = -\frac{(1-\nu^2)}{Eh}x\sin\phi_s\left(P_x + N^T_{,x}\right) \quad , \qquad (4\text{-}2\text{-}5a)$$

$$\frac{1+\nu}{2}u_{,x\theta} + \frac{1}{2x}\left[3-\nu+h_1(1-\nu)\frac{x}{h}\right]u_{,\theta} + \frac{1-\nu}{2}x\sin\phi_s v_{,xx} + \frac{1}{x\sin\phi_s}v_{,\theta\theta} + \frac{1-\nu}{2}\sin\phi_s\left[1+h_1\frac{x}{h}\right]v_{,x}$$

$$-\frac{1-\nu}{2}\frac{\sin\phi_s}{x}\left[1+h_1\frac{x}{h}\right]v + \frac{1}{x\tan\phi_s}w_{,\theta} = -\frac{(1-\nu^2)}{Eh}\left(x\sin\phi_s P_\theta + N^T_{,\theta}\right) \quad , \qquad (4\text{-}2\text{-}5b)$$

$$-\nu\cos\phi_s u_{,x} - \frac{\cos\phi_s}{x}u - \frac{1}{x\tan\phi_s}v_{,\theta} - \frac{\cos^2\phi_s}{x\sin\phi_s}w = -x\sin\phi_s\frac{1-\nu^2}{Eh}\left(P_z - \frac{N^T}{x\tan\phi_s}\right) \quad ,$$

where $h = h_0 + h_1 x$. $\qquad (4\text{-}2\text{-}5c)$

Case 2 Uniform Cone:

$$x\sin\phi_s u_{,xx} + \frac{1-\nu}{2x\sin\phi_s}u_{,\theta\theta} + \sin\phi_s u_{,x} - \frac{\sin\phi_s}{x}u + \frac{1+\nu}{2}v_{,x\theta} - \frac{3-\nu}{2x}v_{,\theta} + \nu\cos\phi_s w_{,x}$$

$$-\frac{\cos\phi_s}{x}w = -\frac{h^2 x\sin\phi_s}{12D}\left(P_x + N^T_{,x}\right) \qquad (4\text{-}2\text{-}6a)$$

$$\frac{1+\nu}{2}u_{,x\theta} + \frac{3-\nu}{2x}u_{,\theta} + x\sin\phi_s\frac{1-\nu}{2}v_{,xx} + \frac{1}{x\sin\phi_s}v_{,\theta\theta} + \sin\phi_s\frac{1-\nu}{2}v_{,x} - \sin\phi_s\frac{1-\nu}{2x}v$$

$$\frac{1}{x\tan\phi_s}w_{,\theta} = -\frac{h^2}{12D}\left(x\sin\phi_s P_\theta + N^T_{,\theta}\right) \qquad (4\text{-}2\text{-}6b)$$

$$-\frac{\nu D\cos^3\phi_s}{k\sin^2\phi_s}u_{,x} - \frac{D\cos^2\phi_s}{xk\sin^2\phi_s}u - \frac{D}{kx\tan^3\phi_s}v_{,\theta} - \frac{D\sin\phi_s}{xk\tan^4\phi_s}w = -x\sin\phi_s\left(P_z - \frac{N^T}{x\tan\phi_s}\right)$$

where $k = \dfrac{h^2}{12\tan^2\phi_s}$ $\qquad (4\text{-}2\text{-}6c)$

Case 3 Uniform Sphere:

$$\sin\phi\, u_{,\phi\phi} + \frac{1-\nu}{2\sin\phi} u_{,\theta\theta} + \cos\phi\, u_{,\phi} - \frac{\cos^2\phi + \nu\sin^2\phi}{\sin\phi} u + \frac{1+\nu}{2} v_{,\phi\theta} - \frac{3-\nu}{2}\cot\phi\, v_{,\theta}$$

$$+ (1+\nu)\sin\phi\, w_{,\phi} = -\frac{h^2}{12D} R\sin\phi\left(R P_\phi + N^T_{,\phi}\right) \quad , \qquad (4\text{-}2\text{-}7a)$$

$$\frac{1+\nu}{2} u_{,\phi\theta} + \frac{3-\nu}{2}\cot\phi\, u_{,\theta} + \frac{1-\nu}{2}\sin\phi\, v_{,\phi\phi} + \frac{1}{\sin\phi} v_{,\theta\theta} + \frac{(1-\nu)}{2}\cos\phi\, v_{,\phi} + \frac{1-\nu}{2}\left(\frac{\sin^2\phi - \cos^2\phi}{\sin\phi}\right) v$$

$$+ (1-\nu) w_{,\theta} = -\frac{h^2 R}{12D}\left(R\sin\phi\, P_\theta + N^T_{,\theta}\right) \quad , \qquad (4\text{-}2\text{-}7b)$$

$$-\frac{\sin\phi}{k}(1+\nu)\frac{D}{R} u_{,\phi} - (1+\nu)\cos\phi\frac{D}{kR^2} u - (1+\nu)\frac{D}{kR^2} v_{,\theta} - 2(1+\nu)\sin\phi\frac{D}{kR^2} w = -R^2\sin\phi\left(P_z - 2\frac{N^T}{R}\right)$$

$$\text{where } k = \frac{h^2}{12R^2} . \qquad (4\text{-}2\text{-}7c)$$

Case 4 Uniform Cylinder:

$$R u_{,xx} + \frac{1-\nu}{2R} u_{,\theta\theta} + \frac{1+\nu}{2} v_{,x\theta} + \nu w_{,x} = -\frac{Rh^2}{12D}\left(P_x + N^T_{,x}\right) , \quad (4\text{-}2\text{-}8a)$$

$$\frac{1+\nu}{2} u_{,x\theta} + \frac{1-\nu}{2} R v_{,xx} + \frac{1}{R} v_{,\theta\theta} + \frac{w_{,\theta}}{R} = -\frac{h^2}{12D}\left(R P_\theta + N^T_{,\theta}\right) , \quad (4\text{-}2\text{-}8b)$$

$$-\frac{\nu D}{kR^2} u_{,x} - \frac{D}{kR^3} v_{,\theta} - \frac{D}{kR^3} w = -R\left(P_z - \frac{N^T}{R}\right) , \quad (4\text{-}2\text{-}8c)$$

$$\text{where } k = \frac{h^2}{12R^2} .$$

4.3 Boundary Conditions

The complete set of governing differential equations (4-2-4) of membrane theory in terms of displacements is of the fourth order while the equations of the general theory is of the eighth order. It has been shown that four boundary conditions are required in the general theory; thus, membrane theory must require two boundary conditions along each edge of the shell since the order of the equation is one half that of the general theory. Referring to Fig. 3-4c it is seen that if all moments and transverse shear are zero the remaining stress resultants are N_ζ and N_η . If the boundary conditions are expressed in terms of stresses they must be given by these two quantities. If the boundary conditions are written in terms of displacements the required conditions must be applied to u_ζ and u_η . Notice that in membrane theory w and ω_η can not be specified at an edge. At a boundary of a membrane shell the force, moment, displacement and rotation vectors are [see equation (3-6-9a-d)]

$$
\begin{aligned}
\underset{\sim}{\bar{N}} &= \underset{\sim}{\bar{a}}_\eta \bar{N}_\eta + \underset{\sim}{\bar{a}}_\zeta \bar{N}_\zeta , \\
\underset{\sim}{\bar{M}} &\equiv 0 , \\
\underset{\sim}{\bar{u}} &= \underset{\sim}{\bar{a}}_\eta \bar{u}_\eta + \underset{\sim}{\bar{a}}_\zeta \bar{u}_\zeta , \\
\underset{\sim}{\bar{\Omega}} &\equiv 0 .
\end{aligned}
\qquad (4\text{-}3\text{-}1a\text{-}d)
$$

From these possible boundary conditions a membrane shell with rigid edge restraint is given by the condition

$$\bar{u}_\eta = \bar{u}_\zeta = 0 .$$

If the shell to be analyzed does not have boundary conditions compatible with those given by equation (4-3-1) the membrane state will be disturbed around the boundary and the general theory should be used.

5. FINITE DIFFERENCE EXPRESSIONS

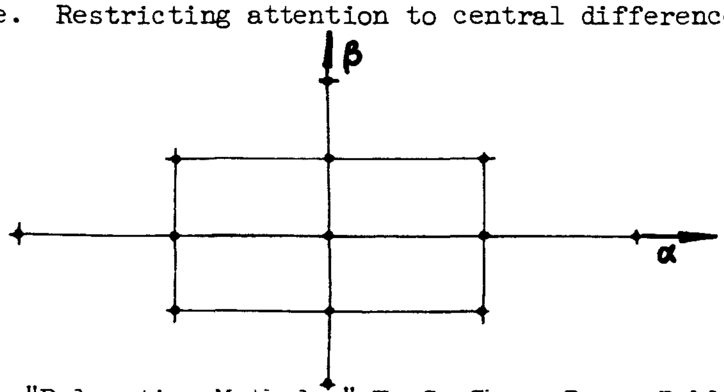
5.1 Introduction

For the solution of the shell equations presented in Chapter 3 finite difference expressions are required for derivatives of the following types:

$$\frac{\partial f(\alpha,\beta)}{\partial\alpha}, \frac{\partial^2 f}{\partial\alpha^2}, \frac{\partial^3 f}{\partial\alpha^3}, \frac{\partial^4 f}{\partial\alpha^4}, \frac{\partial f}{\partial\beta}, \frac{\partial^2 f}{\partial\beta^2}, \frac{\partial^3 f}{\partial\beta^3}, \frac{\partial^4 f}{\partial\beta^4}, \frac{\partial^2 f}{\partial\alpha\partial\beta}, \frac{\partial^3 f}{\partial\alpha^2\partial\beta}, \frac{\partial^3 f}{\partial\alpha\partial\beta^2}, \frac{\partial^4 f}{\partial\alpha^2\partial\beta^2}.$$

The required expressions can be obtained from a truncated Taylor series expansion for a function of two variables, or from an equivalent polynomial expansion. As explained in many standard references*, expressions for forward, backward, or central differences can be obtained through suitable selection of points chosen for the expansion; and, through the truncation of the series, various accuracies can be achieved.

Consider the pattern of mesh points used to represent the function. Since expressions for twelve derivatives are required here, the expansion must encompass a minimum of twelve points in addition to the "origin" about which the expansion is made. Restricting attention to central differences the obvious pattern is:



* For example see "Relaxation Methods," F. S. Shaw, Dover Publications, Inc., 1953

It can be shown that the highest degree of accuracy for a particular number of points results if the pattern "mesh" is at least "rectangular" in the orthogonal coordinate system used. Finite difference expressions involving arbitrary spacings can be derived, as indicated below, but accuracy suffers. However, finite difference expressions involving arbitrary spacings should at least be considered because of the necessity for treating irregular boundaries (boundaries which are not lines of principal curvature in the coordinate system used). Treatment of such boundaries may be accomplished, in principle, by any of several methods such as (1) subdivision of the mesh using a rectangular pattern, (2) use of a pattern composed of arbitrary spacings matching the boundary, or (3) gross approximation using the regular pattern. In general, the third procedure, while the simplest, is inferior in accuracy to either of the first two. Use of the first method requires either an increase in the number of simultaneous equations to be solved or use of an iterative technique. The second method does not necessarily increase the number of simultaneous equations but does require the use of much more complicated difference expressions of somewhat less accuracy than those for the rectangular net.

5.2 Taylor Series Expansion for a Function of Two Variables

The Taylor series for the function $f(\alpha,\beta)$ can be written as:

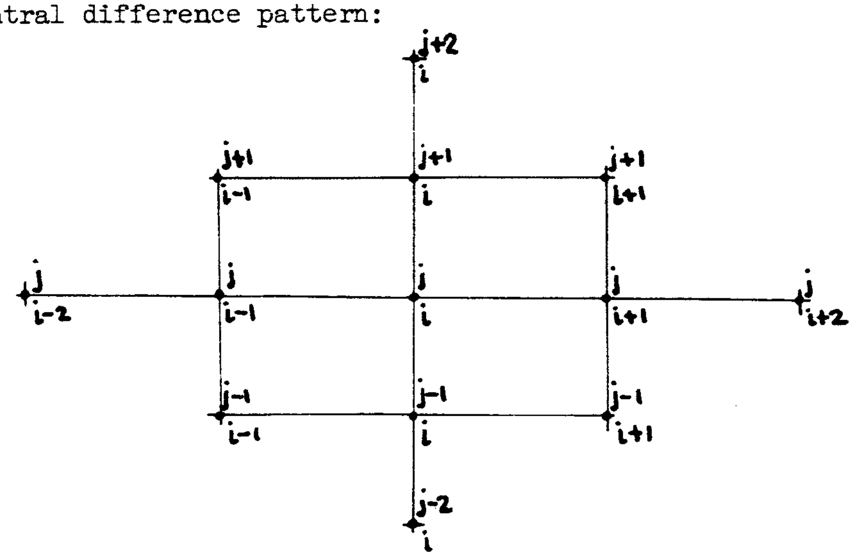
$$f(\alpha,\beta)=f(\alpha_0,\beta_0)+\left[(\alpha-\alpha_0)\partial_\alpha+(\beta-\beta_0)\partial_\beta\right]f(\alpha_0,\beta_0)$$

$$+\frac{1}{2!}\left[(\alpha-\alpha_0)\partial_\alpha+(\beta-\beta_0)\partial_\beta\right]^2 f(\alpha_0,\beta_0)$$

$$+\ldots+\frac{1}{n!}\left[(\alpha-\alpha_0)\partial_\alpha+(\beta-\beta_0)\partial_\beta\right]^n f(\alpha_0,\beta_0) \quad ,$$

where ∂^n_α represents the differential operator $\dfrac{\partial^n}{\partial\alpha^n}$, etc.

Using the central difference pattern:



let:

$$\alpha^j_i=\alpha^0_0 \qquad \beta^j_i=\beta^0_0$$

$$\alpha^{j+1}_i=\alpha^1_0 \qquad \beta^{j+1}_i=\beta^1_0$$

$$\alpha^j_{i+1}=\alpha^0_1 \qquad \beta^j_{i+1}=\beta^0_1$$

etc. etc.

Also let the value of the function $f(\alpha,\beta)$ be represented by f_{ℓ}^{k} at point α_{ℓ}^{k}, β_{ℓ}^{k} (e.g., f_{0}^{2} represents $f(\alpha_{0}^{2},\beta_{0}^{2})$), and denote derivatives at point $(\alpha_{0}^{0}, \beta_{0}^{0})$ as

$$\frac{\partial f}{\partial \alpha} = \partial_{\alpha} f = f_{,\alpha} \quad , \qquad \frac{\partial^2 f}{\partial \alpha^2} = \partial_{\alpha\alpha} f = f_{,\alpha\alpha} \quad ,$$

$$\frac{\partial^2 f}{\partial \alpha \partial \beta} = \partial_{\alpha\beta} f = f_{,\alpha\beta} \quad , \qquad \text{etc.}$$

Using this notation the Taylor series expansion including terms containing desired derivatives through the fourth order is:

$$\begin{aligned} f_{\ell}^{k} = f_{0}^{0} &+ \alpha_{\ell}^{k} f_{,\alpha} + \beta_{\ell}^{k} f_{,\beta} + \frac{\alpha_{\ell}^{2k}}{2} f_{,\alpha\alpha} + \frac{\beta_{\ell}^{2k}}{2} f_{,\beta\beta} + \alpha_{\ell}^{k} \beta_{\ell}^{k} f_{,\alpha\beta} \\ &+ \frac{\alpha_{\ell}^{3k}}{6} f_{,\alpha\alpha\alpha} + \frac{\beta_{\ell}^{3k}}{6} f_{,\beta\beta\beta} + \frac{\alpha_{\ell}^{2k}\beta_{\ell}^{k}}{2} f_{,\alpha\alpha\beta} + \frac{\alpha_{\ell}^{k}\beta_{\ell}^{2k}}{2} f_{,\alpha\beta\beta} \\ &+ \frac{\alpha_{\ell}^{4k}}{24} f_{,\alpha\alpha\alpha\alpha} + \frac{\beta_{\ell}^{4k}}{24} f_{,\beta\beta\beta\beta} + \frac{\alpha_{\ell}^{2k}\beta_{\ell}^{2k}}{4} f_{,\alpha\alpha\beta\beta} \end{aligned} \tag{5-2-2}$$

(It can be seen that this expression is equivalent to the polynomial

$$\begin{aligned} f_{\ell}^{k} = f_{0}^{0} &+ c_1 \alpha_{\ell}^{k} + c_2 \beta_{\ell}^{k} + c_3 \alpha_{\ell}^{2k} + c_4 \beta_{\ell}^{2k} + c_5 \alpha_{\ell}^{k} \beta_{\ell}^{k} + c_6 \alpha_{\ell}^{3k} \\ &+ c_7 \beta_{\ell}^{3k} + c_8 \alpha_{\ell}^{2k} \beta_{\ell}^{k} + c_9 \alpha_{\ell}^{k} \beta_{\ell}^{2k} + c_{10} \alpha_{\ell}^{4k} + c_{11} \beta_{\ell}^{4k} \\ &+ c_{12} \alpha_{\ell}^{2k} \beta_{\ell}^{2k} \Big) . \end{aligned}$$

5.3 Finite Difference Expressions for a Rectangular Pattern

For equal spacing $\bar{h}$ in the coordinate α , and for equal spacing $\bar{k}$ in the coordinate β , then:

$$
\begin{array}{llll}
\alpha_2^0 = 2\bar{h} & \alpha_0^{-1} = 0 & \beta_2^0 = 0 & \beta_0^{-1} = -\bar{k} \\
\alpha_1^0 = \bar{h} & \alpha_0^{-2} = 0 & \beta_1^0 = 0 & \beta_0^{-2} = -2\bar{k} \\
\alpha_{-1}^0 = -\bar{h} & \alpha_1^1 = \bar{h} & \beta_{-1}^0 = 0 & \beta_1^1 = \bar{k} \\
\alpha_{-2}^0 = -2\bar{h} & \alpha_{-1}^1 = -\bar{h} & \beta_{-2}^0 = 0 & \beta_{-1}^1 = \bar{k} \\
\alpha_0^2 = 0 & \alpha_1^{-1} = \bar{h} & \beta_0^2 = 2\bar{k} & \beta_1^{-1} = -\bar{k} \\
\alpha_0^1 = 0 & \alpha_{-1}^{-1} = -\bar{h} & \beta_0^1 = \bar{k} & \beta_{-1}^{-1} = -\bar{k}
\end{array}
$$

So that the matrix equation is from (5.2.2):

$$
\begin{bmatrix}
2\bar{h} & 2\bar{h}^2 & \frac{4\bar{h}^3}{3} & \frac{2\bar{h}^4}{3} & & & & & & & & \\
\bar{h} & \frac{\bar{h}^2}{2} & \frac{\bar{h}^3}{6} & \frac{\bar{h}^4}{24} & & & & & & & & \\
-\bar{h} & \frac{\bar{h}^2}{2} & -\frac{\bar{h}^3}{6} & \frac{\bar{h}^4}{24} & & & & & & & & \\
-2\bar{h} & 2\bar{h}^2 & -\frac{4\bar{h}^3}{3} & \frac{2\bar{h}^4}{3} & & & & & & & & \\
& & & & 2\bar{k} & 2\bar{k}^2 & \frac{4\bar{k}^3}{3} & \frac{2\bar{k}^4}{3} & & & & \\
& & & & \bar{k} & \frac{\bar{k}^2}{2} & \frac{\bar{k}^3}{6} & \frac{\bar{k}^4}{24} & & & & \\
& & & & -\bar{k} & \frac{\bar{k}^2}{2} & -\frac{\bar{k}^3}{6} & \frac{\bar{k}^4}{24} & & & & \\
& & & & -2\bar{k} & 2\bar{k}^2 & -\frac{4\bar{k}^3}{3} & \frac{2\bar{k}^4}{3} & & & & \\
\bar{h} & \frac{\bar{h}^2}{2} & \frac{\bar{h}^3}{6} & \frac{\bar{h}^4}{24} & \bar{k} & \bar{k}^2 & \bar{k}^3 & \bar{k}^4 & \bar{h}\bar{k} & \frac{\bar{h}^2\bar{k}}{2} & \frac{\bar{h}\bar{k}^2}{2} & \frac{\bar{h}^2\bar{k}^2}{4} \\
-\bar{h} & \frac{\bar{h}^2}{2} & -\frac{\bar{h}^3}{6} & \frac{\bar{h}^4}{24} & \bar{k} & \bar{k}^2 & \bar{k}^3 & \bar{k}^4 & -\bar{h}\bar{k} & \frac{\bar{h}^2\bar{k}}{2} & -\frac{\bar{h}\bar{k}^2}{2} & \frac{\bar{h}^2\bar{k}^2}{4} \\
\bar{h} & \frac{\bar{h}^2}{2} & \frac{\bar{h}^3}{6} & \frac{\bar{h}^4}{24} & -\bar{k} & \bar{k}^2 & -\bar{k}^3 & \bar{k}^4 & -\bar{h}\bar{k} & -\frac{\bar{h}^2\bar{k}}{2} & \frac{\bar{h}\bar{k}^2}{2} & \frac{\bar{h}^2\bar{k}^2}{4} \\
-\bar{h} & \frac{\bar{h}^2}{2} & -\frac{\bar{h}^3}{6} & \frac{\bar{h}^4}{24} & -\bar{k} & \bar{k}^2 & -\bar{k}^3 & \bar{k}^4 & \bar{h}\bar{k} & -\frac{\bar{h}^2\bar{k}}{2} & -\frac{\bar{h}\bar{k}^2}{2} & \frac{\bar{h}^2\bar{k}^2}{4}
\end{bmatrix}
\begin{bmatrix}
f_{,\alpha} \\ f_{,\alpha\alpha} \\ f_{,\alpha\alpha\alpha} \\ f_{,\alpha\alpha\alpha\alpha} \\ f_{,\beta} \\ f_{,\beta\beta} \\ f_{,\beta\beta\beta} \\ f_{,\beta\beta\beta\beta} \\ f_{,\alpha\beta} \\ f_{,\alpha\alpha\beta} \\ f_{,\alpha\beta\beta} \\ f_{,\alpha\alpha\beta\beta}
\end{bmatrix}
=
\begin{bmatrix}
f_2^0 - f_0^0 \\ f_1^0 - f_0^0 \\ f_{-1}^0 - f_0^0 \\ f_{-2}^0 - f_0^0 \\ f_0^2 - f_0^0 \\ f_0^1 - f_0^0 \\ f_0^{-1} - f_0^0 \\ f_0^{-2} - f_0^0 \\ f_1^1 - f_0^0 \\ f_{-1}^1 - f_0^0 \\ f_1^{-1} - f_0^0 \\ f_{-1}^{-1} - f_0^0
\end{bmatrix}
$$

The inverse equation is:

$$
\begin{bmatrix} f_{,\alpha} \\ f_{,\alpha\alpha} \\ f_{,\alpha\alpha\alpha} \\ f_{,\alpha\alpha\alpha\alpha} \\ f_{,\beta} \\ f_{,\beta\beta} \\ f_{,\beta\beta\beta} \\ f_{,\beta\beta\beta\beta} \\ f_{,\alpha\beta} \\ f_{,\alpha\alpha\beta} \\ f_{,\alpha\beta\beta} \\ f_{,\alpha\alpha\beta\beta} \end{bmatrix}
=
\begin{bmatrix}
-\frac{1}{12h} & \frac{8}{12h} & -\frac{8}{12h} & \frac{1}{12h} & & & & & & & & \\
-\frac{1}{12h^2} & \frac{16}{12h^2} & \frac{16}{12h^2} & -\frac{1}{12h^2} & & & & & & & & \\
\frac{1}{2h^3} & -\frac{2}{2h^3} & \frac{2}{2h^3} & -\frac{1}{2h^3} & & & & & & & & \\
\frac{1}{h^4} & -\frac{4}{h^4} & -\frac{4}{h^4} & \frac{1}{h^4} & & & & & & & & \\
& & & & -\frac{1}{12k} & \frac{8}{12k} & -\frac{8}{12k} & \frac{1}{12k} & & & & \\
& & & & -\frac{1}{12k^2} & \frac{16}{12k^2} & \frac{16}{12k^2} & -\frac{1}{12k^2} & & & & \\
& & & & \frac{1}{2k^3} & -\frac{2}{2k^3} & \frac{2}{2k^3} & -\frac{1}{2k^3} & & & & \\
& & & & \frac{1}{k^4} & -\frac{4}{k^4} & -\frac{4}{k^4} & \frac{1}{k^4} & & & & \\
& & & & & & & & \frac{1}{4hk} & -\frac{1}{4hk} & -\frac{1}{4hk} & \frac{1}{4hk} \\
& & & & & -\frac{2}{2h^2k} & \frac{2}{2h^2k} & & \frac{1}{2h^2k} & \frac{1}{2h^2k} & -\frac{1}{2h^2k} & -\frac{1}{2h^2k} \\
& -\frac{2}{2hk^2} & \frac{2}{2hk^2} & & & & & & \frac{1}{2hk^2} & -\frac{1}{2hk^2} & \frac{1}{2hk^2} & -\frac{1}{2hk^2} \\
& -\frac{2}{h^2k^2} & -\frac{2}{h^2k^2} & & & -\frac{2}{h^2k^2} & -\frac{2}{h^2k^2} & & \frac{1}{h^2k^2} & \frac{1}{h^2k^2} & \frac{1}{h^2k^2} & \frac{1}{h^2k^2}
\end{bmatrix}
\begin{bmatrix} f_2^0 - f_0^0 \\ f_1^0 - f_0^0 \\ f_{-1}^0 - f_0^0 \\ f_{-2}^0 - f_0^0 \\ f_0^2 - f_0^0 \\ f_0^1 - f_0^0 \\ f_0^{-1} - f_0^0 \\ f_0^{-2} - f_0^0 \\ f_1^1 - f_0^0 \\ f_{-1}^1 - f_0^0 \\ f_1^{-1} - f_0^0 \\ f_{-1}^{-1} - f_0^0 \end{bmatrix}
$$

(5-3-2)

From which the desired finite difference expressions can be extracted as:

$$
\begin{aligned}
f_{,\alpha} &= \frac{\partial f}{\partial \alpha} = \frac{1}{12\bar{h}}\left[-f_{2}^{0} + 8f_{1}^{0} - 8f_{-1}^{0} + f_{-2}^{0}\right] \\
f_{,\alpha\alpha} &= \frac{\partial^2 f}{\partial \alpha^2} = \frac{1}{12\bar{h}^2}\left[-f_{2}^{0} + 16f_{1}^{0} - 30f_{0}^{0} + 16f_{-1}^{0} - f_{-2}^{0}\right] \\
f_{,\alpha\alpha\alpha} &= \frac{\partial^3 f}{\partial \alpha^3} = \frac{1}{2\bar{h}^3}\left[f_{2}^{0} - 2f_{1}^{0} + 2f_{-1}^{0} - f_{-2}^{0}\right] \\
f_{,\alpha\alpha\alpha\alpha} &= \frac{\partial^4 f}{\partial \alpha^4} = \frac{1}{\bar{h}^4}\left[f_{2}^{0} - 4f_{1}^{0} + 6f_{0}^{0} - 4f_{-1}^{0} + f_{-2}^{0}\right] \\
f_{,\beta} &= \frac{\partial f}{\partial \beta} = \frac{1}{12\bar{k}}\left[-f_{0}^{2} + 8f_{0}^{1} - 8f_{0}^{-1} + f_{0}^{-2}\right] \\
f_{,\beta\beta} &= \frac{\partial^2 f}{\partial \beta^2} = \frac{1}{12\bar{k}^2}\left[-f_{0}^{2} + 16f_{0}^{1} - 30f_{0}^{0} + 16f_{0}^{-1} - f_{0}^{-2}\right] \\
f_{,\beta\beta\beta} &= \frac{\partial^3 f}{\partial \beta^3} = \frac{1}{2\bar{k}^3}\left[f_{0}^{2} - 2f_{0}^{1} + 2f_{0}^{-1} - f_{0}^{-2}\right] \\
f_{,\beta\beta\beta\beta} &= \frac{\partial^4 f}{\partial \beta^4} = \frac{1}{\bar{k}^4}\left[f_{0}^{2} - 4f_{0}^{1} + 6f_{0}^{0} - 4f_{0}^{-1} + f_{0}^{-2}\right] \\
f_{,\alpha\beta} &= \frac{\partial^2 f}{\partial \alpha \partial \beta} = \frac{1}{4\bar{h}\bar{k}}\left[f_{1}^{1} - f_{-1}^{1} - f_{1}^{-1} + f_{-1}^{-1}\right] \\
f_{,\alpha\alpha\beta} &= \frac{\partial^3 f}{\partial \alpha^2 \partial \beta} = \frac{1}{2\bar{h}^2\bar{k}}\left[-2f_{0}^{1} + 2f_{0}^{-1} + f_{1}^{1} + f_{-1}^{1} - f_{1}^{-1} - f_{-1}^{-1}\right] \\
f_{,\alpha\beta\beta} &= \frac{\partial^3 f}{\partial \alpha \partial \beta^2} = \frac{1}{2\bar{h}\bar{k}^2}\left[-2f_{1}^{0} + 2f_{-1}^{0} + f_{1}^{1} - f_{-1}^{1} + f_{1}^{-1} - f_{-1}^{-1}\right] \\
f_{,\alpha\alpha\beta\beta} &= \frac{\partial^4 f}{\partial \alpha^2 \partial \beta^2} = \frac{1}{\bar{h}^2\bar{k}^2}\left[-2f_{1}^{0} - 2f_{-1}^{0} - 2f_{0}^{1} - 2f_{0}^{-1} + f_{1}^{1} + f_{-1}^{1} + f_{1}^{-1} + f_{-1}^{-1} + 4f_{0}^{0}\right]
\end{aligned}
\qquad (5\text{-}3\text{-}3\ a\text{-}\ell)
$$

The same finite difference relationships expressed in pattern form are:

$$\frac{\partial}{\partial \alpha_{ij}} = \frac{1}{12h}\left[\begin{array}{ccccc} +1 & -8 & 0 & +8 & -1 \end{array}\right]$$

$$\frac{\partial^2}{\partial \alpha_{ij}^2} = \frac{1}{12h^2}\left[\begin{array}{ccccc} -1 & +16 & -30 & +16 & -1 \end{array}\right]$$

$$\frac{\partial^3}{\partial \alpha_{ij}^3} = \frac{1}{2h^3}\left[\begin{array}{ccccc} -1 & +2 & 0 & -2 & +1 \end{array}\right]$$

$$\frac{\partial^4}{\partial \alpha_{ij}^4} = \frac{1}{h^4}\left[\begin{array}{ccccc} +1 & -4 & +6 & -4 & +1 \end{array}\right]$$

$$\frac{\partial}{\partial \beta_{ij}} = \frac{1}{12k}\left[\begin{array}{c} -1 \\ +8 \\ 0 \\ -8 \\ +1 \end{array}\right] \qquad \frac{\partial^2}{\partial \beta_{ij}^2} = \frac{1}{12k^2}\left[\begin{array}{c} -1 \\ 16 \\ -30 \\ 16 \\ -1 \end{array}\right] \qquad \frac{\partial^3}{\partial \beta_{ij}^3} = \frac{1}{2k^3}\left[\begin{array}{c} +1 \\ -2 \\ 0 \\ 2 \\ -1 \end{array}\right] \qquad \frac{\partial^4}{\partial \beta_{ij}^4} = \frac{1}{k^4}\left[\begin{array}{c} +1 \\ -4 \\ +6 \\ -4 \\ +1 \end{array}\right]$$

$$\frac{\partial^2}{\partial \alpha \partial \beta_{ij}} = \frac{1}{4hk}\left[\begin{array}{ccc} -1 & & +1 \\ & i,j & \\ +1 & & -1 \end{array}\right] \qquad \frac{\partial^3}{\partial \alpha^2 \partial \beta_{ij}} = \frac{1}{2h^2k}\left[\begin{array}{ccc} +1 & -2 & +1 \\ & i,j & \\ -1 & +2 & -1 \end{array}\right]$$

$$\frac{\partial^3}{\partial \alpha \partial \beta_{ij}^2} = \frac{1}{2hk^2}\left[\begin{array}{ccc} -1 & & +1 \\ +2 & i,j & -2 \\ -1 & & +1 \end{array}\right] \qquad \frac{\partial^4}{\partial \alpha^2 \partial \beta_{ij}^2} = \frac{1}{h^2k^2}\left[\begin{array}{ccc} +1 & -2 & +1 \\ -2 & +4 & -2 \\ +1 & -2 & +1 \end{array}\right]$$

It should be noted that by truncating the Taylor series expansion to include only second order terms simplified expressions (less accurate than those presented above) can be obtained for the first and second "total" derivatives. These are:

$$
\begin{aligned}
f_{,\alpha} &= \frac{\partial f}{\partial \alpha} = \frac{1}{2\bar{h}}\left(f_{1}^{0} - f_{-1}^{0}\right) \\
f_{,\alpha\alpha} &= \frac{\partial^2 f}{\partial \alpha^2} = \frac{1}{\bar{h}^2}\left(f_{1}^{0} - 2f_{0}^{0} + f_{-1}^{0}\right) \\
f_{,\beta} &= \frac{\partial f}{\partial \beta} = \frac{1}{2\bar{k}}\left(f_{0}^{1} - f_{0}^{-1}\right) \\
f_{,\beta\beta} &= \frac{\partial^2 f}{\partial \beta^2} = \frac{1}{\bar{k}^2}\left(f_{0}^{1} - 2f_{0}^{0} + f_{0}^{-1}\right)
\end{aligned}
\qquad (5\text{-}3\text{-}4\ a\text{-}d)
$$

The finite difference expressions for a "square" mesh can, of course, be obtained by setting $\bar{h}$ equal to $\bar{k}$ in the relationships presented here. If this is done the expressions become identical to those given by Shaw*.

*Loc. cit.

5.4 Finite Difference Expressions for Arbitrary Spacing

If the point pattern used in the last section is retained but the spacing is considered to be arbitrary, then the general matrix equation is:

$$
\begin{bmatrix}
\alpha_2^0 & \frac{(\alpha_2^0)^2}{2} & \frac{(\alpha_2^0)^3}{6} & \frac{(\alpha_2^0)^4}{24} & & & & & & & & \\
\alpha_1^0 & \frac{(\alpha_1^0)^2}{2} & \frac{(\alpha_1^0)^3}{6} & \frac{(\alpha_1^0)^4}{24} & & & & & & & & \\
\alpha_{-1}^0 & \frac{(\alpha_{-1}^0)^2}{2} & \frac{(\alpha_{-1}^0)^3}{6} & \frac{(\alpha_{-1}^0)^4}{24} & & & & & & & & \\
\alpha_{-2}^0 & \frac{(\alpha_{-2}^0)^2}{2} & \frac{(\alpha_{-2}^0)^3}{6} & \frac{(\alpha_{-2}^0)^4}{24} & & & & & & & & \\
 & & & & \beta_0^2 & \frac{(\beta_0^2)^2}{2} & \frac{(\beta_0^2)^3}{6} & \frac{(\beta_0^2)^4}{24} & & & & \\
 & & & & \beta_0^1 & \frac{(\beta_0^1)^2}{2} & \frac{(\beta_0^1)^3}{6} & \frac{(\beta_0^1)^4}{24} & & & & \\
 & & & & \beta_0^{-1} & \frac{(\beta_0^{-1})^2}{2} & \frac{(\beta_0^{-1})^3}{6} & \frac{(\beta_0^{-1})^4}{24} & & & & \\
 & & & & \beta_0^{-2} & \frac{(\beta_0^{-2})^2}{2} & \frac{(\beta_0^{-2})^3}{6} & \frac{(\beta_0^{-2})^4}{24} & & & & \\
\alpha_1^1 & \frac{(\alpha_1^1)^2}{2} & \frac{(\alpha_1^1)^3}{6} & \frac{(\alpha_1^1)^4}{24} & \beta_1^1 & \frac{(\beta_1^1)^2}{2} & \frac{(\beta_1^1)^3}{6} & \frac{(\beta_1^1)^4}{24} & \alpha_1^1\beta_1^1 & \frac{(\alpha_1^1)^2\beta_1^1}{2} & \frac{\alpha_1^1(\beta_1^1)^2}{2} & \frac{(\alpha_1^1)^2(\beta_1^1)^2}{4} \\
\alpha_{-1}^1 & \frac{(\alpha_{-1}^1)^2}{2} & \frac{(\alpha_{-1}^1)^3}{6} & \frac{(\alpha_{-1}^1)^4}{24} & \beta_{-1}^1 & \frac{(\beta_{-1}^1)^2}{2} & \frac{(\beta_{-1}^1)^3}{6} & \frac{(\beta_{-1}^1)^4}{24} & \alpha_{-1}^1\beta_{-1}^1 & \frac{(\alpha_{-1}^1)^2\beta_{-1}^1}{2} & \frac{\alpha_{-1}^1(\beta_{-1}^1)^2}{2} & \frac{(\alpha_{-1}^1)^2(\beta_{-1}^1)^2}{4} \\
\alpha_1^{-1} & \frac{(\alpha_1^{-1})^2}{2} & \frac{(\alpha_1^{-1})^3}{6} & \frac{(\alpha_1^{-1})^4}{24} & \beta_1^{-1} & \frac{(\beta_1^{-1})^2}{2} & \frac{(\beta_1^{-1})^3}{6} & \frac{(\beta_1^{-1})^4}{24} & \alpha_1^{-1}\beta_1^{-1} & \frac{(\alpha_1^{-1})^2\beta_1^{-1}}{2} & \frac{\alpha_1^{-1}(\beta_1^{-1})^2}{2} & \frac{(\alpha_1^{-1})^2(\beta_1^{-1})^2}{4} \\
\alpha_{-1}^{-1} & \frac{(\alpha_{-1}^{-1})^2}{2} & \frac{(\alpha_{-1}^{-1})^3}{6} & \frac{(\alpha_{-1}^{-1})^4}{24} & \beta_{-1}^{-1} & \frac{(\beta_{-1}^{-1})^2}{2} & \frac{(\beta_{-1}^{-1})^3}{6} & \frac{(\beta_{-1}^{-1})^4}{24} & \alpha_{-1}^{-1}\beta_{-1}^{-1} & \frac{(\alpha_{-1}^{-1})^2\beta_{-1}^{-1}}{2} & \frac{\alpha_{-1}^{-1}(\beta_{-1}^{-1})^2}{2} & \frac{(\alpha_{-1}^{-1})^2(\beta_{-1}^{-1})^2}{4}
\end{bmatrix}
\begin{bmatrix}
f_{,\alpha} \\ f_{,\alpha\alpha} \\ f_{,\alpha\alpha\alpha} \\ f_{,\alpha\alpha\alpha\alpha} \\ f_{,\beta} \\ f_{,\beta\beta} \\ f_{,\beta\beta\beta} \\ f_{,\beta\beta\beta\beta} \\ f_{,\alpha\beta} \\ f_{,\alpha\alpha\beta} \\ f_{,\alpha\beta\beta} \\ f_{,\alpha\alpha\beta\beta}
\end{bmatrix}
=
\begin{bmatrix}
f_2^0 - f_0^0 \\ f_1^0 - f_0^0 \\ f_{-1}^0 - f_0^0 \\ f_{-2}^0 - f_0^0 \\ f_0^2 - f_0^0 \\ f_0^1 - f_0^0 \\ f_0^{-1} - f_0^0 \\ f_0^{-2} - f_0^0 \\ f_1^1 - f_0^0 \\ f_{-1}^1 - f_0^0 \\ f_1^{-1} - f_0^0 \\ f_{-1}^{-1} - f_0^0
\end{bmatrix}
$$

The desired expressions for the derivatives can be stated explicitly in terms of the values of the function at the various points if the inverse of the square matrix is available. However, the manual inversion of this matrix would be a tedious task since in general the coordinates of each point must be retained as algebraic quantities. Use of finite difference expressions for arbitrary spacing of the mesh points perhaps could be accomplished most efficiently in conjunction with the general computer solution of the shell problem through use of a subroutine which would receive coordinates of points in the mesh from the controlling routine, substitute them in the matrix equation above, perform the inversion numerically, and return the desired coefficients in the analogous finite difference expression to the controlling routine for insertion into the governing equation, following which the combined coefficients would be summed for each point to yield the shell equation for the given central point in difference form. By repeating the process for each central point, the required set of shell difference equations could be formed.

6. ANALYSIS OF MULTICELLULAR SHELL STRUCTURES

6.1 General Considerations

The problem considered in this chapter is the evaluation of elastic stresses and deformations in multicellular shell structures as shown in Fig. 6-1. This structure is subjected to certain static loadings such as internal pressure, body forces, and temperature gradients. If it is assumed that these loads are symmetric with respect to the diametrical planes bisecting each pair of cells located opposite each other, only one-half of a cell [Fig. 6-2] need be considered in an analysis. Each cell on the other hand is composed of four basic structural elements, namely: flat plate, segmented cone, sphere, and cylinder as seen in Fig. 6-3. Referring to Fig. 6-3, it is noted that the edge rotation (ω_η) along the boundary line $\overline{ghijkl}$ is zero because of the assumed symmetric loads.

The method of analysis to be presented for the solution of this complex shell structure is similar to that of the "slope-deflection"[1] method used in the analysis of indeterminate space structures and the "direct stiffness"[2] method used in the analysis of idealized stiffened shell structures. An advantage of this method is that it is readily adapted to solution by high-speed digital computer. To place the structure in a form to which this method can be applied, it is "cut" along boundary lines governed by the geometry; i.e., boundary of cone, sphere, etc. as seen in Fig. 6-3. Then it is necessary to solve each

[1]Tsui, E. Y. W., "Analysis of Haunched Octagonal Girder Space Frames," Journal of the Structural Div., ASCE Proceedings Vol. 85, No. ST 6, June 1959

[2]Turner, M. J., Clough, R. W., Martin, H. C., and Topp, L. J., "Stiffness and Deflection Analysis of Complex Structures," J.A.S., Vol 23, Sept 1956

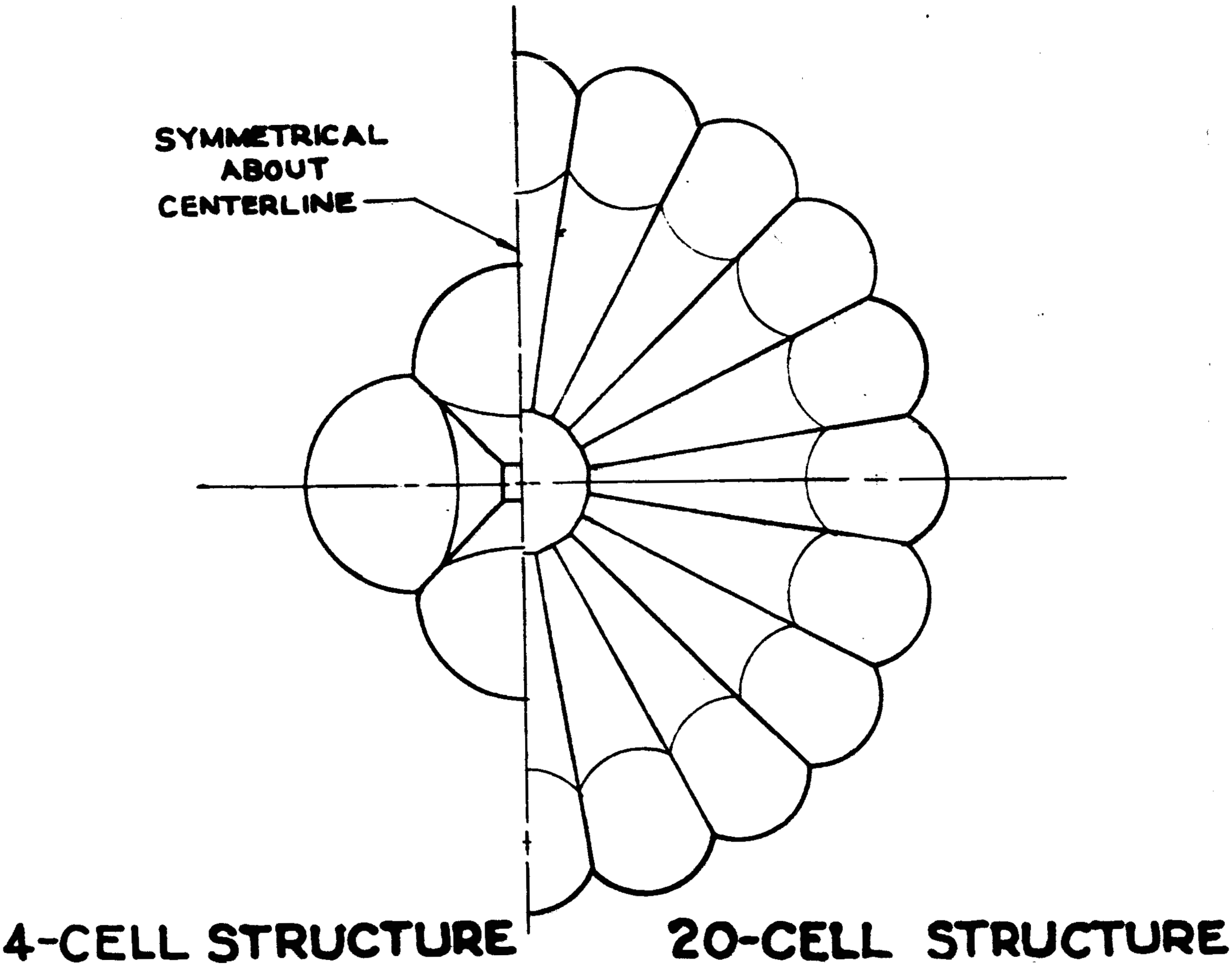


FIG. 6-1
MULTICELLULAR SHELL STRUCTURE

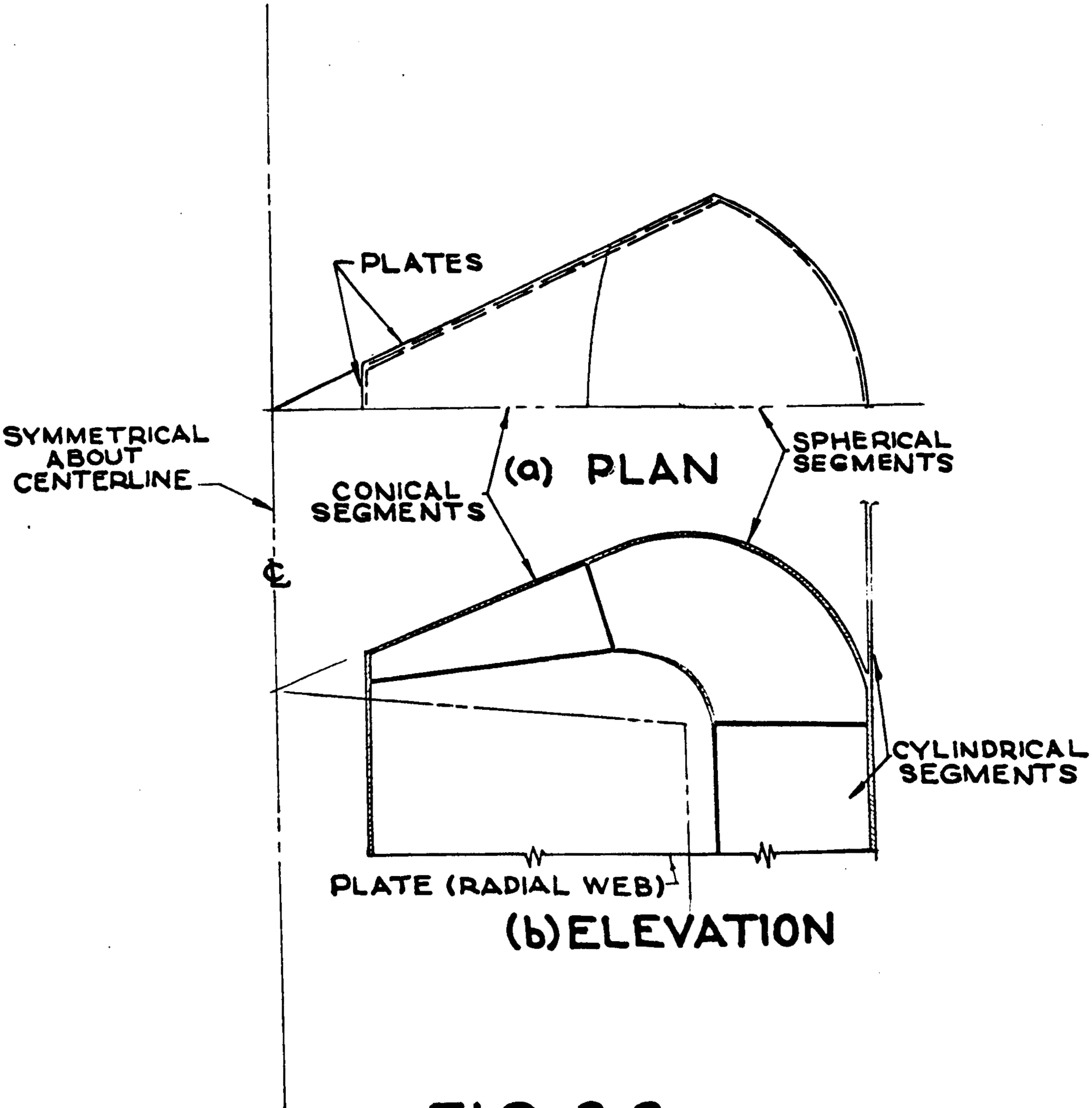


FIG. 6-2
HALF CELL MODEL

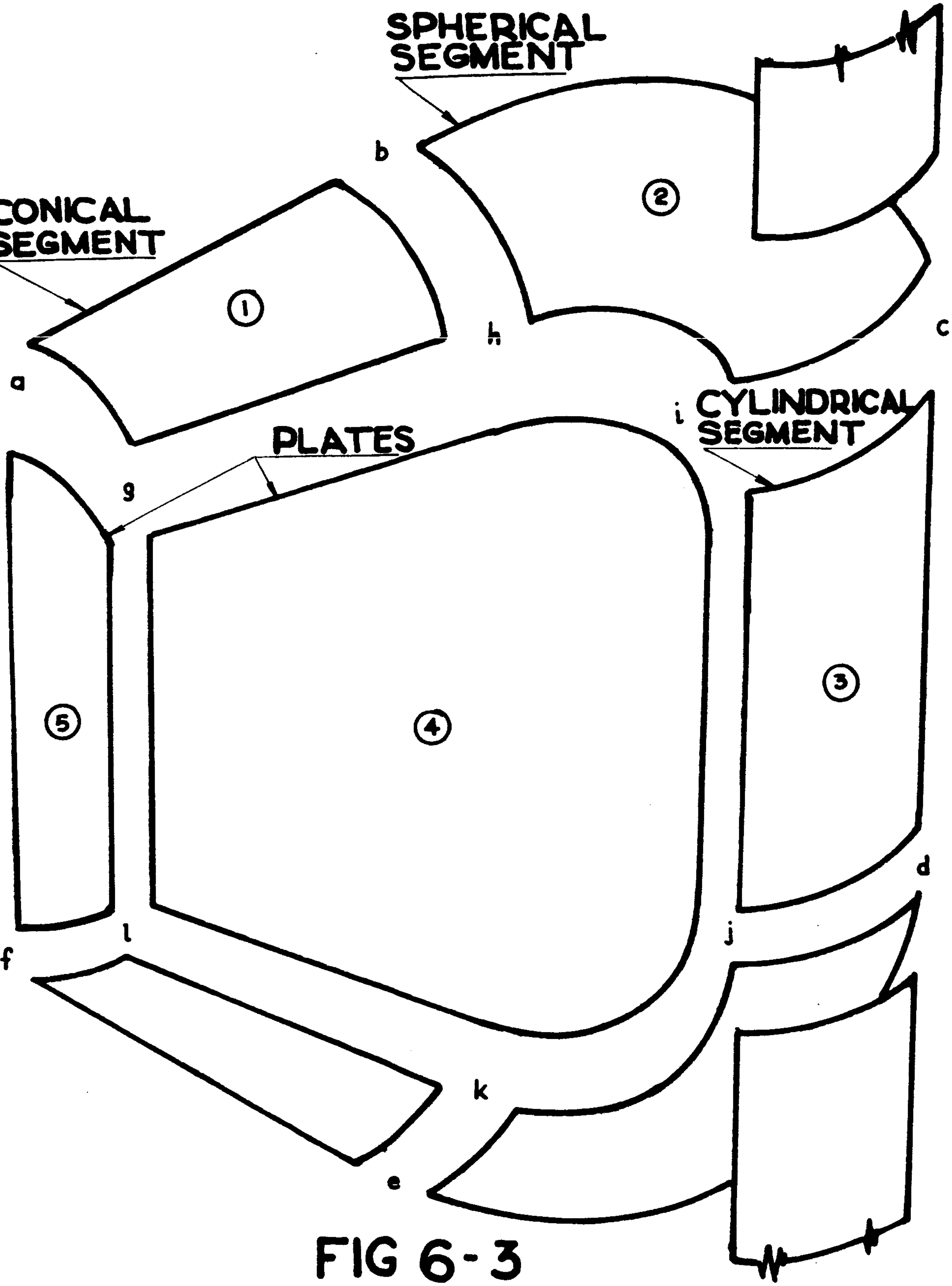


FIG 6-3
BASIC SHELL ELEMENTS
OF MULTICELLULAR STRUCTURE

of the individual shell segments separately under the action of intermediate and edge loads and then join the segments together to form a continuous structure by means of applying proper boundary loads or displacements. It is assumed that the cut structure will be made continuous at a finite number of discrete points along the cut boundaries. The general scheme is to develop a "stiffness matrix" for each shell segment which relates boundary forces to boundary displacements of the shell segments. Then at the juncture of any two or more segments the forces must be in equilibrium and the displacements must be compatible. At those points which are not juncture points but are boundaries of the structure either the forces or displacements must be known. From this given information a set of simultaneous equations can be developed to solve for the unknown displacements at a finite number of points. These equations are expressed in terms of the stiffness coefficients of the individual shell elements, known boundary forces and displacements, and boundary forces due to the intermediate loads. After the unknown displacements are computed the forces at the discrete points are obtained through use of the stiffness matrix. This then yields forces and displacements along the cut boundaries which produce a continuous structure. Once the boundary conditions are known stresses and displacements internal to the shell segments can be computed.

6.2 Stress and Deformation of Shell Segments Under Intermediate Loads

As a first step in the overall analysis of a multicellular shell structure it is required to obtain a solution for stresses and deformations of the shell segments [Fig. 6-3] with their edges fixed from displacement and rotation subjected to intermediate loads such as pressure, thermal gradients, etc. The three governing differential equations in terms of the displacement components,

u , v , w for all of the shell segments are given by equation (3-4-18) and the coefficients of this equation are given in Table 3-5-2. When the shell segments are of uniform thickness the governing equations are given by equations (3-5-2) through (3-5-5). Let $L[u , v , w]$ denote the linear operator of the governing equations and $g(\alpha , \beta)$ the right hand side which is a function of the intermediate loads. The independent variables α , β are orthogonal curvilinear coordinates and for the specific shell segments these variables are given in Table 3-5-1. Then instead of writing equation (3-4-18) or (3-5-2) through (3-5-5) in full the governing equations are given by

$$L[u , v , w] = g(\alpha , \beta) \qquad (6\text{-}2\text{-}1)$$

For a plate the equations uncouple so that it is required to solve two sets of equations; i.e., one for u , v and the other for w . At the shell boundary we require that the displacement and rotation vectors be zero $[\underset{\sim}{\bar{u}} = \underset{\sim}{\bar{\Omega}} = 0]$. This implies, as seen in section 3.6, that the displacement components and normal derivation of w are zero

$$u = v = w = \frac{\partial w}{\partial \zeta} = 0 \qquad (6\text{-}2\text{-}2)$$

The boundary curve in terms of the specific shell coordinates can be obtained by the methods described in Appendix I. These are shown in Fig. 6-4 for a given set of dimensions R , R_1 , R_0 , φ_1 and φ_2 . Symmetry is implied across the centerline due to the loading and geometry. Because of the complex boundary curve of some of the shell segments it is felt that a solution of the partial differential equations (6-2-1) with boundary conditions (6-2-2) should be obtained by finite-difference methods.

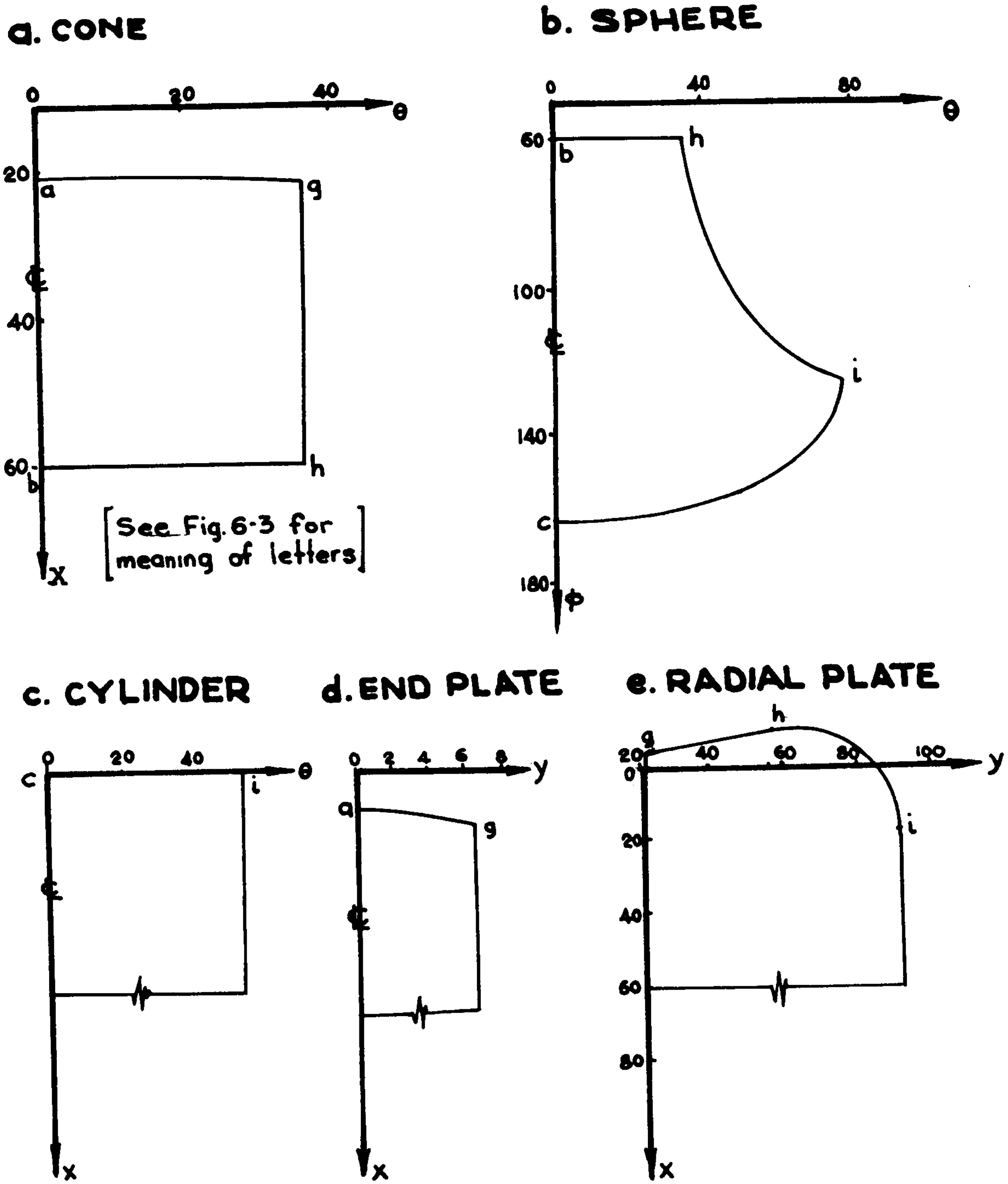


FIG. 6-4
BOUNDARY CURVES OF SEGMENTED SHELLS FORMING THE MULTICELLULAR SHELL STRUCTURE

The scheme in finite-difference methods is to replace the continuous problem by one having a finite number of variables. To accomplish this discretization the independent variables (α , β) are replaced by a number of points which may be determined by the intersection of a finite number of "mesh" lines. Thus a rectangular mesh such as is shown in Fig. 6-5 produces a finite number of "mesh points" (pivotal points, nodal points) such as A . Some other meshes which fill the entire space are square, triangular, and hexagonal.

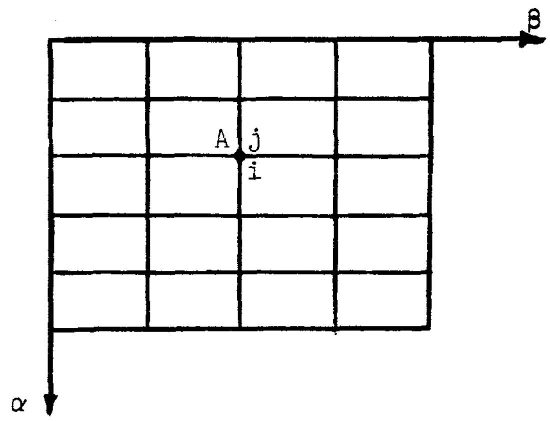


Fig. 6-5 Rectangular Finite-Difference Mesh

Now, each of the dependent variables $u(\alpha , \beta)$, $v(\alpha , \beta)$, $w(\alpha , \beta)$ is replaced by the variables u_i^j , v_i^j , w_i^j defined only at the mesh points (i , j) . If finite-difference expressions for derivatives of the dependent variables are developed [see Chapter 5] for a given mesh type the governing equation (6-2-1) can be replaced by an algebraic equation (this implies 3 equations) at a mesh point in terms of u_i^j , v_i^j , w_i^j at a specific number of mesh points. Then an algebraic equation can be written at each mesh point inside and on the boundary curve. In all cases a sufficient number of equations can be written for the unknowns. Hence, this finite difference method replaces equations (6-2-1) and (6-2-2) by a set of simultaneous equations

$$AX = B \tag{6-2-3}$$

where X is a column vector of dependent variables (u_i^j, v_i^j, w_i^j), B is a column vector of intermediate loads and A is a square matrix.

As an example of writing the governing differential equation in difference form we consider a uniform thickness cylinder. The governing equations are given by equation (3-5-4) as

$$a_1 u,_{xx} + a_2 u,_{\theta\theta} + a_3 v,_{x\theta} + a_4 w,_x = -\frac{Rh^2}{12D}(p_x + N,_x^T)$$

$$b_1 u,_{x\theta} + b_2 v,_{xx} + b_3 v,_{\theta\theta} + b_4 w,_{xx\theta} + b_5 w,_{\theta\theta\theta} + b_6 w,_\theta = -\frac{h^2}{12D}\left(Rp_\theta + N,_\theta^T + \frac{M,_\theta^T}{R}\right)$$

$$c_1 u,_x + c_2 v,_{xx\theta} + c_3 v,_{\theta\theta\theta} + c_4 v,_\theta + c_5 w,_{xxx} + c_6 w,_{xx\theta\theta}$$
$$+ c_7 w,_{\theta\theta\theta} + c_8 w = -R\left(p_z - \frac{N^T}{R} + M,_{xx}^T + \frac{M,_{\theta\theta}^T}{R^2}\right)$$

For the cylinder let the boundary curve be given as θ = constant, x = constant and two lines of symmetry as shown in Fig. 6-6. This domain is now covered by a uniform rectangular mesh which has mesh lines that coincide

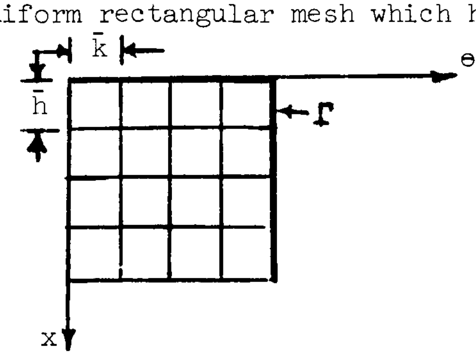


Fig. 6-6 Rectangular Mesh for Cylindrical Shell Segment

with the boundary curve Γ . For this particular mesh the derivatives are given by equation (5-3-3). In order to write a sufficient number of equations for the unknowns it is necessary that the lower order equations such as given by equation (5-3-4) be used. It was found that for this particular problem if lower order second derivatives and a lower order third derivative of v in θ as given by

$$\frac{\partial^3 v}{\partial \theta^3} \approx \frac{1}{k^3}\left[v_0^1 - 3v_0^0 + 3v_0^{-1} - v_0^{-2}\right] \tag{6-2-5}$$

were used, the resulting difference equations are valid at all mesh points, and that a sufficient number of equations were obtained for the unknowns. These difference equations are:

$$A_1[u_1^0+u_{-1}^0] + A_2 u_0^0 + A_3[u_0^1+u_0^{-1}] + A_4[v_1^1-v_{-1}^1-v_1^{-1}+v_{-1}^{-1}] + A_5[w_1^0-w_{-1}^0] = A_0^0$$

$$B_1[u_1^1-u_{-1}^1-u_1^{-1}+u_{-1}^{-1}] + B_2[v_1^0+v_{-1}^0] + B_3 v_0^0 + B_4[v_0^1+v_0^{-1}] + B_5[w_0^1-w_0^{-1}]$$
$$+B_6[w_1^1-w_{-1}^1-w_1^{-1}-w_{-1}^{-1}] + B_7[w_0^2-w_0^{-2}] = B_0^0 \qquad \text{(6-2-6a-c)}$$

$$C_1[u_1^0-u_{-1}^0] + C_2\, v_0^{-2} + 3C_2 v_0^0 + C_3 v_0^1 + C_4 v_0^{-1} + C_5[v_1^1+v_{-1}^1-v_1^{-1}-v_{-1}^{-1}] + C_6[w_2^0+w_{-2}^0]$$
$$+C_7[w_1^0+w_{-1}^0] + C_8[w_0^1+w_0^{-1}] + C_9[w_1^1+w_{-1}^1+w_1^{-1}+w_{-1}^{-1}] + C_{10}[w_0^2+w_0^{-2}] + C_{11} w_0^0$$
$$= C_0^0$$

where

$$A_1 = R/\bar{h}^2$$

$$A_3 = (1-\nu)/(2R\bar{k}^2)$$

$$A_2 = -2(A_1 + A_3)$$

$$A_4 = (1+\nu)/(8\bar{h}\bar{k})$$

$$A_5 = \nu/2\bar{h}$$

$$A_0^* = -Rh^2(P_x + N^T_{,x})/(12D)$$

$$k = h^2/(12R^2)$$

$$B_1 = (1+\nu)/(8\bar{h}\bar{k}) = A_4$$

$$B_2 = (1-\nu)R(1+4k)/(2\bar{h}^2)$$

$$B_4 = (1+k)/(R\bar{k}^2)$$

$$B_3 = -2[B_2 + B_4]$$

$$B_6 = -(2-\nu)kR/(2\bar{h}^2\bar{k})$$

$$B_7 = -k/(2R\bar{k}^3)$$

$$B_5 = -2[B_6 + B_7] + 1/(2R\bar{k})$$

$$B_0^* = -h^2(Rp_\theta + N^T_{,\theta} + M^T_{,\theta}/R)/(12D)$$

$$C_1 = -\nu D/(2kR^2\bar{h})$$

$$C_2 = -D/(R^3\bar{k}^3)$$

$$C_3 = -C_2 - D/(2R^3k\bar{k}) - (2-\nu)D/(R\bar{h}^2\bar{k})$$

$$C_4 = -C_3 - 4C_2$$

$$C_5 = (2-\nu)D/(2R\bar{h}^2\bar{k})$$

$$C_6 = -RD/\bar{h}^4$$

$$C_7 = 4D/(R\bar{h}^2\bar{k}^2) - 4C_6$$

$$C_8 = 4D/(R^3\bar{k}^4) + 4D/(R\bar{h}^2\bar{k}^2)$$

$$C_9 = -2D/(R\bar{h}^2\bar{k}^2)$$

$$C_{10} = -D/(R^3\bar{k}^4)$$

$$C_{11} = 6C_{10} + 6C_6 + 4C_9 - D/(R^3 k)$$

$$C_0^* = -R\left(p_z - N^T/R + M^T_{,xx} + M^T_{,\theta\theta}/R^2\right)$$

Although the finite-difference method is simple in principle the discretization gives rise to a number of practical problems such as graded nets, curved boundaries, and the solution of a large set of simultaneous equations. These have been discussed by Forsythe[1], Collatz[2], and Varga[3]. The formulation and solution of equation (6-2-3) is perhaps the most important and time consuming part of the analysis and is discussed in Appendix IV.

Once the displacements are known the internal stress resultants and couples can be computed by equations (3-4-12, 13, 14 and 16). Along the shell boundary three components of stress resultants and a stress couple as given by equations (3-6-7a-d) can be computed at each mesh point. These quantities are denoted as the fixed edge forces F^f which are given by the column vector

$$F^f = \begin{Bmatrix} \bar{N}_\eta \\ \bar{N}_\zeta \\ \bar{Q} \\ \bar{M}_\zeta \end{Bmatrix} \qquad (6\text{-}2\text{-}7)$$

---

[1]Milne, W. E., Numerical Solution of Differential Equations, John Wiley & Sons, New York, 1957

[2]Collatz, L., The Numerical Treatment of Differential Equations, Springer-Verlag, Berlin, 1960

[3]Varga, R. S., Matrix Iterative Analysis, Prentice-Hall, Inc., New York, 1962

This equation will be used directly in the complete analysis and enters in section 6.3.

The preceding presents a method to obtain a solution of shell segments with their edges fixed, due to intermediate loads.

6.3 Stress and Deformation of Shell Segments due to Edge Effects

To apply the slope-deflection method it is required to determine a stiffness matrix for each shell segment. Let points on the boundary of a shell be denoted by $i = 1, 2, 3 -- n$ as shown in Fig. 6-7. At each point there are [see section 3.6] four boundary forces F_i and four displacement components

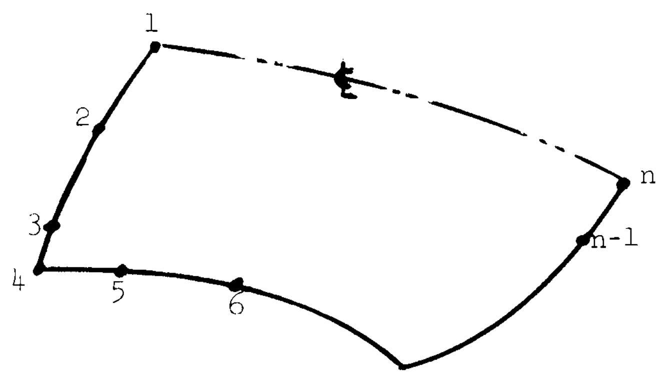


Fig. 6-7 Shell Segment

δ_i which represent respectively

$$F_i = \begin{Bmatrix} \bar{N}_\eta \\ \bar{N}_\zeta \\ \bar{Q} \\ \bar{M}_\zeta \end{Bmatrix} , \quad \delta_i = \begin{Bmatrix} \bar{u}_\eta \\ \bar{u}_\zeta \\ \bar{w} \\ \bar{\omega}_\eta \end{Bmatrix} . \qquad (6\text{-}3\text{-}1a,b)$$

For a perfectly elastic shell segment if a unit displacement $(u_\eta)_i$ is introduced at the boundary points 1 , 2 , 3 -- n , the induced force $\bar{N}_\eta$ at point 1 due to these displacements can be expressed as

$$\left(\bar{N}_\eta\right)_1 = k_{11}(\bar{u}_\eta)_1 + k_{12}(\bar{u}_\eta)_2 -- k_{1n}(\bar{u}_\eta)_n = \sum_{j=1}^{n} k_{1j}(\bar{u}_\eta)_j \quad ,$$

where k_{ij} are the stiffness influence coefficients. In general for a given shell segment we can write the constitutive edge relations in matrix from as

$$F = k\delta \quad . \tag{6-3-2}$$

The coefficients of k must be computed by methods such as the finite-difference method given in section 6.2. A procedure to obtain the coefficients of k follows. One displacement component is given a unit value while the others are taken to be zero. This unit displacement is applied at each boundary point i.e., $i = 1 , 2 , 3$ -- n in succession and the resulting boundary forces are computed at each of the boundary points. The stiffness influence coefficients are then equal to the respective force values.

When the shell is acted upon by intermediate loads additional forces must be added to equation (6-3-2), that is, the forces obtained from fixed edge support [Eq. (6-2-7)]. Then equation (6-3-2) becomes

$$F = k\delta + F^f \quad . \tag{6-3-3}$$

It is noted that the inverse of the stiffness matrix k is called the flexibility matrix "a" and the elements of "a" are called the flexibility influence coefficients (a_{ij}) .

6.4 Equilibrium Conditions and Compatibility Requirements at the Juncture of Shell Segments

When two or more shell segments are connected and are required to form an integral structure, continuity requirements of the boundary forces F and displacements δ must be satisfied at the juncture. From section 3.6 on boundary conditions the requirements for two shells are

$$\underset{\sim}{\bar{N}}_1 = \underset{\sim}{\bar{N}}_2 \quad , \quad \underset{\sim}{\bar{u}}_1 = \underset{\sim}{\bar{u}}_2 \quad ,$$

$$\underset{\sim}{\bar{M}}_1 = \underset{\sim}{\bar{M}}_2 \quad , \quad \underset{\sim}{\bar{\Omega}}_1 = \underset{\sim}{\bar{\Omega}}_2 \quad , \qquad (6\text{-}4\text{-}1a\text{-}d)$$

where $\underset{\sim}{\bar{N}}$ and $\underset{\sim}{\bar{u}}$ have three components each while $\underset{\sim}{\bar{M}}$ and $\underset{\sim}{\bar{\Omega}}$ have only one. For a general shell structure it is sometimes desirable to decompose the forces and displacements along rectangular coordinate directions so that the components of shells 1 and 2, for instance, are taken in the same direction. To this end, we can write

$$\underset{\sim}{\bar{N}} = i\bar{N}_x + j\bar{N}_y + k\bar{Q}_z = \underset{\sim}{\bar{a}}_\eta \bar{N}_\eta + \underset{\sim}{\bar{a}}_\zeta \bar{N}_\zeta + \underset{\sim}{a}_3 \bar{Q} \quad ,$$

then

$$\bar{N}_x = i \cdot \underset{\sim}{\bar{N}} = i \cdot \underset{\sim}{\bar{a}}_\eta \bar{N}_\eta + i \cdot \underset{\sim}{\bar{a}}_\zeta \bar{N}_\zeta + i \cdot \underset{\sim}{a}_3 \bar{Q} \quad ,$$

$$\bar{N}_y = j \cdot \underset{\sim}{\bar{N}} = j \cdot \underset{\sim}{\bar{a}}_\eta \bar{N}_\eta + j \cdot \underset{\sim}{\bar{a}}_\zeta \bar{N}_\zeta + j \cdot \underset{\sim}{a}_3 \bar{Q} \quad , \qquad (6\text{-}4\text{-}2a\text{-}c)$$

$$\bar{N}_z = k \cdot \underset{\sim}{\bar{N}} = k \cdot \underset{\sim}{a}_\eta \bar{N}_\eta + k \cdot \underset{\sim}{a}_\zeta \bar{N}_\zeta + k \cdot \underset{\sim}{a}_3 \bar{Q} \quad ,$$

and the displacement

$$\underset{\sim}{\bar{u}} = i\bar{u}_x + j\bar{u}_y + k\bar{u}_z = \underset{\sim}{\bar{a}}_\eta \bar{u}_\eta + \underset{\sim}{\bar{a}}_\zeta \bar{u}_\zeta + \underset{\sim}{a}_3 \bar{w} \quad ,$$

has components

$$\bar{u}_x = i\cdot\underset{\sim}{\bar{u}} = i\cdot\underset{\sim}{\bar{a}}_\eta \bar{u}_\eta + j\cdot\underset{\sim}{\bar{a}}_\zeta \bar{u}_\zeta + \underset{\sim}{a}_3 \bar{w} \quad ,$$

$$\bar{u}_y = j\cdot\underset{\sim}{\bar{u}} = i\cdot\underset{\sim}{\bar{a}}_\eta \bar{u}_\eta + j\cdot\underset{\sim}{\bar{a}}_\zeta \bar{u}_\zeta + \underset{\sim}{a}_3 \bar{w} \quad , \qquad (6\text{-}4\text{-}3a\text{-}c)$$

$$\bar{u}_z = k\cdot\underset{\sim}{\bar{u}} = i\cdot\underset{\sim}{\bar{a}}_\eta \bar{u}_\eta + j\cdot\underset{\sim}{\bar{a}}_\zeta \bar{u}_\zeta + \underset{\sim}{a}_3 \bar{w} \quad .$$

Then, equations (6-3-1a,b) can be rewritten as

$$F_i = \begin{Bmatrix} \bar{N}_x \\ \bar{N}_y \\ \bar{N}_z \\ \bar{M}_\zeta \end{Bmatrix} \quad , \qquad \delta_i = \begin{Bmatrix} \bar{u}_x \\ \bar{u}_y \\ \bar{u}_z \\ \bar{\omega}_\eta \end{Bmatrix} \quad . \qquad (6\text{-}4\text{-}4a,b)$$

When the slope-deflection equations are employed the sign of the forces and displacements are such that the vector sum of forces at a point are zero and the displacement components are equal. Thus if points 1 , 3 , and 5 form a common juncture A , and F_i , δ_i denote the force and displacement vector in equations (6-4-4a,b) at a point i , then the equilibrium and continuity requirements at point A are

$$\vec{F}_1 + \vec{F}_3 + \vec{F}_5 = 0$$

$$\delta_1 = \delta_3 = \delta_5 \qquad (6\text{-}4\text{-}5a,b)$$

Note that F and δ are column vectors and equation (6-4-5a) denotes four equations and equation (6-4-5b) denotes eight equations.

6.5 General Procedure of Analysis

At this stage it is assumed that stiffness matrices have been obtained for all shell segments employed in a given structure and that it is possible to obtain the boundary forces F^f due to intermediate loads. It is required to obtain a solution of the entire structure under the prescribed intermediate loading and boundary conditions.

The solution proceeds as follows:

1) A finite number of points are chosen along the cut shell boundaries at which continuity will be satisfied.

2) From equilibrium requirements, equations of the type of equation (6-4-5a) are written at each juncture point. At points on a boundary which are not juncture points δ or F or components thereof must be given. This yields a number of simultaneous linear equations in terms of unknown forces and displacements.

3) The unknown forces are related to displacements and fixed edge forces by means of the stiffness matrix [equation (6-3-3)]. Then by means of the continuity of displacements [equation (6-4-5b)] a set of algebraic equations is obtained in the unknown displacements

$$K\delta^R = \overline{F} \tag{6-5-1}$$

where δ^R are the redundant displacements, K the stiffness matrix of the entire structure, and $\overline{F}$ is a function of known data.

4) Once a solution of equation (6-5-1) is obtained for δ^R , the forces at the juncture points are found by equation (6-3-3). This in effect solves the entire problem since solutions internal to each shell segment can be obtained once the boundary conditions are given.

7. CONCLUSIONS AND RECOMMENDATIONS

7.1 Conclusions

As a result of an extensive literature search, it is observed that no analytic procedure exists that could be applied directly in the evaluation of the stress fields peculiar to the multi-cell juncture configuration as specified by NASA/MSFC. Consequently, it is believed, at least for the present time, that the most feasible method of handling such a problem is to solve it numerically by means of high-speed digital computers.

The necessary investigations required to formulate the theory and a numerical method for the prediction of the membrane and bending stresses and the corresponding deformations of multicellular shell structures have been performed. The method of solution is presented in terms of loads, geometry and material properties.

It should be pointed out, however, that the resultant method of analysis considers only the elastic theory of stresses as it applies to thin shell structures. Thermal stresses due to temperature gradients have been taken into account and orthotropic plate and cylindrical shell elements have also been considered.

In view of the general aspects of residual stresses in welded structures (Appendix III), it is felt that the joint efficiency and attenuation length of welds should be incorporated into the theoretical analysis through the experimental determination of residual stresses.

In order to ensure the feasibility of the numerical procedure developed in the present work, a direct method of solving large systems of simultaneous equations has also been developed (Appendix IV). Check problems involving 1700 equations have been solved by this method in the IBM 7094. Run time for such a problem is about sixteen minutes using double precision arithmetic. This accomplishment exceeds expectations at the start of the investigation, and indicates that the present procedure is practicable for the solution of stress problems in multicellular shell structures.

7.2 Recommendations for Further Investigations

Based on the experience and results obtained in the present work, it is recommended that the following additional investigations should be made:

1. Development of workable digital programs for the stresses and deformations of the specific structures using both isotropic and orthotropic shell segments. This work shall include: (a) flow charts of the developed programs, (b) optimization of mesh size and total computer running time, presumably using the available method of solving large sets of equations recently developed by LMSC, (c) numerical examples to illustrate how the analytic procedure is carried out, especially in the process of evaluating the influence coefficients of shell segments as well as satisfying the compatibility conditions along the shell junctures.

2. Experimental determination of the pattern of residual stresses for certain specific welded connections to be used in the structures

under consideration, using the same type of material and welding process as for the prototype structure. This work should also include the testing of large samples of specimens and the application of the results to determine the joint efficiencies of welds through a justified theory of failure.

3. Conduct investigations on the effects of various possible means of stress-relief on large welded structures.

4. Using the present available formulation, evaluate the buckling loads of both isotropic and orthotropic segmental cylindrical shells with displacements satisfying appropriate boundary conditions.

5. Feasibility study of predicting the dynamic response of the bulkhead elements due to certain simple forcing functions, including the approximate determination of the natural frequencies of shell segments. This can be done numerically by means of the available equations developed in the present work with pertinent inertia terms added to the corresponding equations of equilibrium.

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Appendix I

I. GEOMETRY OF THE BULKHEAD

In this appendix the geometry of the bulkhead structure is completely defined. This includes the necessary dimensions, coordinate systems, and intersection of the component shells which comprise the bulkhead.

I.1 Dimensions of Bulkhead

Referring to Fig. I-1, the geometry of the bulkhead structure can be completely determined if the following five quantities are given:

$$R \ , \ R_1 \ , \ R_0 \ , \ \varphi_1 \ , \text{ and } \ \varphi_2 \ .$$

For particular bulkheads these quantities are restricted in the following manner:

$$\varphi_1 = \frac{\pi}{N} \ , \text{ where } \ N = \text{number of cells} = 4 \ , 6 \ , 8 \text{ --- } 20$$

$$\frac{R}{R_1 - R} \geq \tan \varphi_1$$

The true dimensions which will be frequently referred to in the development of the numerical analysis are:

$$\overline{ab} = (R_1 - R) \cot \varphi_2$$

$$\overline{bc} = (R_1 - R) \csc \varphi_2$$

$$\varphi_3 = \sin^{-1}\left[\frac{\sin \varphi_2}{\left(\frac{R_1}{R} - 1\right)}\right]$$

$$\overline{bd} = \overline{be} = \overline{bc} \cos \varphi_3$$

$$\varphi_4 = \frac{\pi}{2} - \varphi_3$$

$$\overline{ah} = (R_1 - R) \cos \varphi_1$$

$$\overline{ch} = (R_1 - R) \sin \varphi_1$$

$$\varphi_5 = \cos^{-1} \left(\frac{\overline{ch}}{R} \right)$$

$$\varphi_6 = \frac{\pi}{2} - \varphi_5 + \varphi_1$$

$$R_T = R \sin \varphi_5$$

$$\overline{ag} = \overline{ah} + R_T$$

$$\overline{bi} = R_0 \sec (\varphi_2 + \varphi_3)$$

$$\overline{bj} = R_0 \operatorname{cosec} \varphi_2 \qquad \text{(I-1-1a-m)}$$

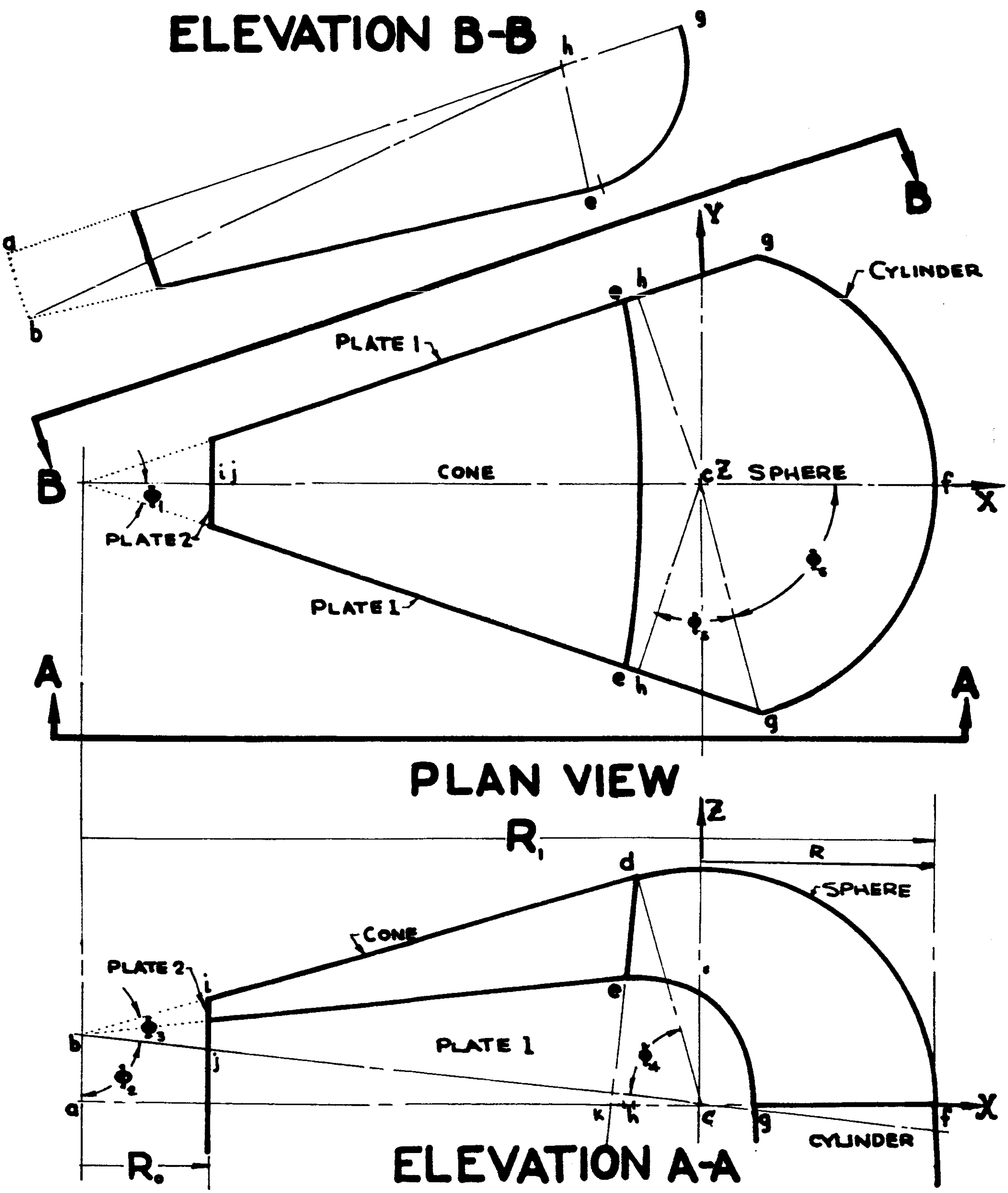


FIG I-1. BULKHEAD GEOMETRY

1.2 Geometry and Coordinate Systems for Specific Shell Surfaces

The equations for a sphere, cone, cylinder and plate are given for a system of rectangular coordinates (X, Y, Z). Then a system of orthogonal curvilinear coordinates are presented for each of the shell elements. From this information the first fundamental form, equation (3-2-3), and the base vectors, equation (3-2-4), of the surface are given. By use of the second fundamental form, equation (3-2-10), the principal radii of curvature are obtained.

A. Geometry of Sphere

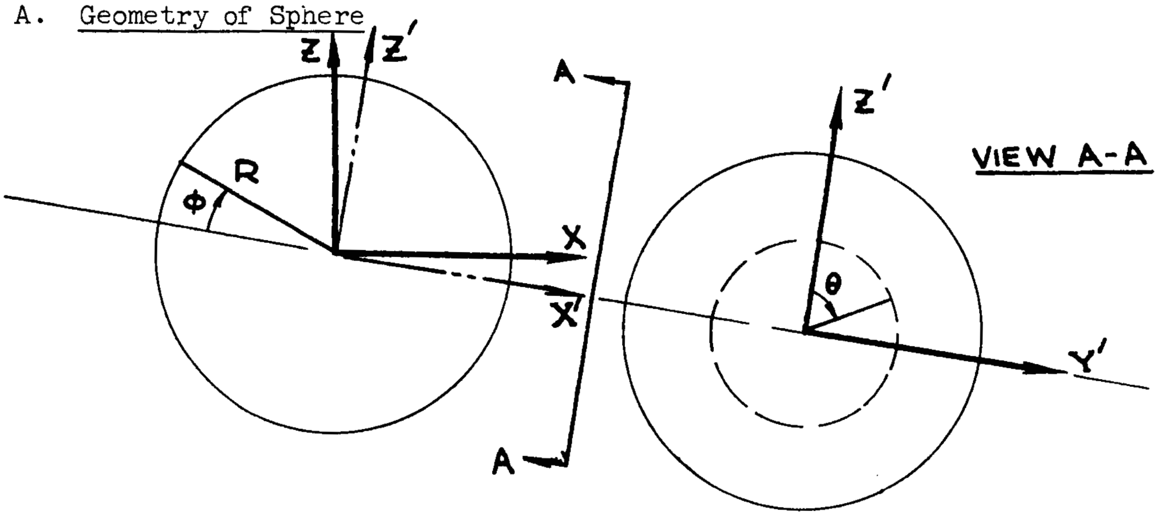


Fig. I-2 Geometry of Sphere

A.1 Equation of Sphere

The equation of a sphere in the rectangular coordinate system X, Y, Z is

$$f_s = X^2 + Y^2 + Z^2 - R^2 = 0 \quad . \qquad \text{(I-2-1)}$$

A.2 Curvilinear Coordinates

If the curvilinear coordinates $X^1 = \varphi$, $X^2 = \theta$ defined by Fig. I-2 are introduced, then the relation between X, Y, Z and φ, θ is

$$X = -R\cos\phi\sin\phi_2 + R\sin\phi\cos\phi_2\cos\theta \quad ,$$

$$Y = R\sin\phi\sin\theta \quad , \qquad \text{(I-2-2a-c)}$$

$$Z = R\cos\phi\cos\phi_2 + R\sin\phi\cos\theta\sin\phi_2 \quad .$$

A.3 First Fundamental Form

From equation (3-2-3) and equation (3-4-1) the first fundamental form of the surface of the sphere can be written as

$$(ds)^2 = d\underset{\sim}{\bar{r}}\cdot d\underset{\sim}{\bar{r}} = A_1^2(d\phi)^2 + 2A_1A_2\cos\chi\, d\phi\, d\theta + A_2^2(d\theta)^2 \quad , \qquad \text{(I-2-3)}$$

where

$$A_1^2 = \underset{\sim}{\bar{r}}_{,\phi}\cdot\underset{\sim}{\bar{r}}_{,\phi} \quad ,$$

$$A_2^2 = \underset{\sim}{\bar{r}}_{,\theta}\cdot\underset{\sim}{\bar{r}}_{,\theta} \quad ,$$

$$A_1A_2\cos\chi = \underset{\sim}{\bar{r}}_{,\phi}\cdot\underset{\sim}{\bar{r}}_{,\theta} \quad .$$

If the unit base vectors of the X , Y , Z , coordinate system are designated i , j , k , then the radius vector $\underset{\sim}{\bar{r}}$ is

$$\underset{\sim}{\bar{r}} = Xi + Yj + Zk \quad . \qquad \text{(I-2-4)}$$

By equation (I-2-2), equation (I-2-4) can be written in terms of θ and φ as

$$\underset{\sim}{\bar{r}} = R\Big\{\big(-\cos\phi\sin\phi_2 + \sin\phi\cos\phi_2\cos\theta\big)i + \big(\sin\phi\sin\theta\big)j$$
$$+\big(\cos\phi\cos\phi_2 + \sin\phi\sin\phi_2\cos\theta\big)k\Big\} \quad . \qquad \text{(I-2-5)}$$

The derivative of $\bar{\underset{\sim}{r}}$ with respect to θ and φ are

$$\bar{\underset{\sim}{r}}_{,\phi} = R\Big\{\left(\sin\phi\sin\phi_2 + \cos\phi\cos\phi_2\cos\theta\right)i + \left(\cos\phi\sin\theta\right)j$$
$$+\left(-\sin\phi\cos\phi_2 + \cos\phi\sin\phi_2\cos\theta\right)k\Big\} \qquad , \qquad \text{(I-2-6a-b)}$$
$$\bar{\underset{\sim}{r}}_{,\theta} = R\left\{-\sin\phi\cos\phi_2\sin\theta\, i + \sin\phi\cos\theta\, j - \sin\phi\sin\phi_2\sin\theta\, k\right\} .$$

Then the Lamé parameters A_1 , A_2 are

$$A_1 = R \quad ,$$
$$\qquad\qquad \text{(I-2-7a-b)}$$
$$A_2 = R\sin\phi \ ,$$

and, since $\bar{\underset{\sim}{r}}_{,\theta}\cdot\bar{\underset{\sim}{r}}_{,\phi} = 0$; θ , φ form an orthogonal system of curvilinear coordinates.

The unit base vectors of the middle-surface are

$$\bar{\underset{\sim}{a}}_1 = \frac{a_1}{A_1} = \frac{\bar{\underset{\sim}{r}}_{,\phi}}{A_1} = \left(\sin\phi\sin\phi_2 + \cos\phi\cos\phi_2\cos\theta\right)i + \left(\cos\phi\sin\theta\right)j + \left(-\sin\phi\cos\phi_2 + \cos\phi\sin\phi_2\cos\theta\right)k ,$$
$$\bar{\underset{\sim}{a}}_2 = \frac{a_2}{A_2} = \frac{\bar{\underset{\sim}{r}}_{,\theta}}{A_2} = -\sin\theta\cos\phi_2\, i + \cos\theta\, j - \sin\phi_2\sin\theta\, k \quad , \qquad \text{(I-2-8a-c)}$$
$$\underset{\sim}{a}_3 = \bar{\underset{\sim}{a}}_1 \times \bar{\underset{\sim}{a}}_2 = \left(-\cos\phi\sin\phi_2 + \sin\phi\cos\phi_2\cos\theta\right)i + \left(\sin\phi\sin\theta\right)j + \left(\cos\phi\cos\phi_2 + \sin\phi\sin\phi_2\cos\theta\right)k .$$

A.4 Second Fundamental Form

By use of the second fundamental form equation (3-2-10) it is possible to obtain the principal radii of curvature as given by equation (3-4-1b) when the coordinates coincide with lines of principal curvature. The radii r_1 and r_2 can then be found, and it can be proved that the coordinates are lines of principal curvature by the following argument

$$b_{11} = -\frac{A_1^2}{r_1} = \underset{\sim}{a}_3 \cdot \bar{\underset{\sim}{a}}_{1_{,1}} \quad ,$$

$$b_{12} = b_{21} = 0 = \underset{\sim}{a}_3 \cdot \bar{\underset{\sim}{a}}_{1_{,2}} = -\underset{\sim}{a}_3 \cdot \bar{\underset{\sim}{a}}_{2_{,1}} \quad , \qquad \text{(I-2-9 a-c)}$$

$$b_{22} = -\frac{A_2^2}{r} = \underset{\sim}{a}_3 \cdot \bar{\underset{\sim}{a}}_{2_{,2}} \quad .$$

After performing the above scalar products it can be seen that the principal radii of curvature are

$$r_1 = r_2 = R \quad , \qquad \text{(I-2-10)}$$

and that $b_{12} = b_{21} = 0$.

B. Geometry of Cone

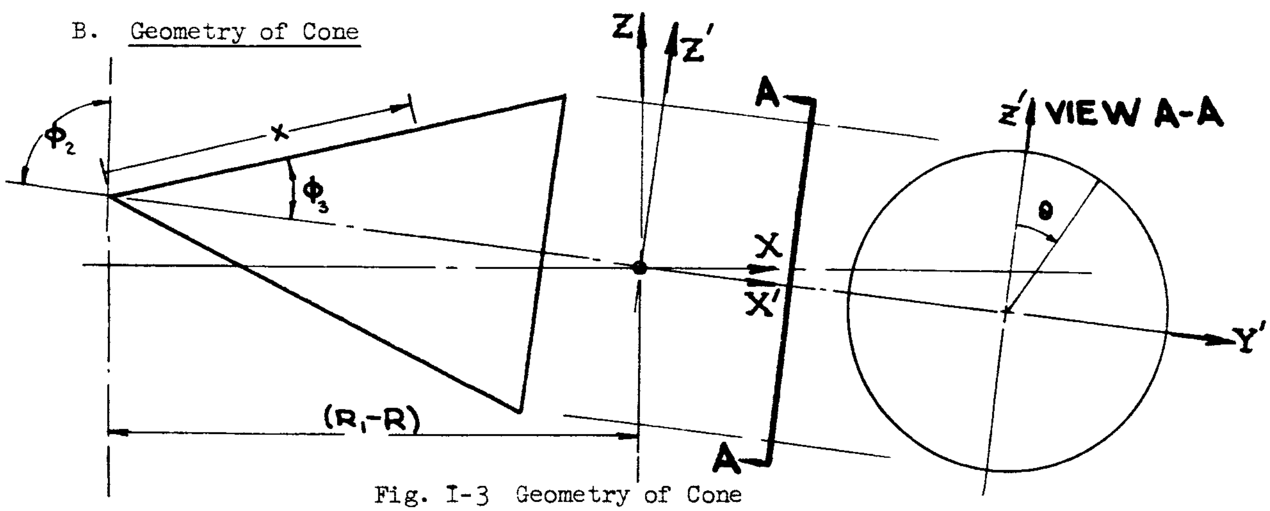


Fig. I-3 Geometry of Cone

B.1 Equation of Cone

The equation of a cone in the rectangular coordinate system X' , Y' , Z' is

$$f_c = (Y')^2 + (Z')^2 - \tan^2\phi_3 \left[X' + \frac{(R_1-R)}{\sin\phi_2}\right]^2 = 0 \quad . \qquad \text{(I-2-11)}$$

By a rotation of coordinates through the angle ϕ_2 the equation of a cone in the rectangular coordinate system X , Y , Z is

$$f_c = Y^2 + (X\cos\phi_2 + Z\sin\phi_2)^2 - \tan^2\phi_3\left[X\sin\phi_2 - Z\cos\phi_2 + \frac{R_1 - R}{\sin\phi_2}\right]^2 = 0 \;. \quad (I-2-12)$$

B.2 <u>Curvilinear Coordinates</u>

If the curvilinear coordinates $X^1 = x$; $X^2 = \theta$ shown in Fig. I-3 are introduced, then the relation between X , Y , Z and x , θ is

$$X = \left[x\cos\phi_3 - \frac{R_1 - R}{\sin\phi_2}\right]\sin\phi_2 + \left[x\sin\phi_3\cos\theta\right]\cos\phi_2 \;,$$

$$Y = x\sin\phi_3\sin\theta \;, \qquad (I-2-13a-c)$$

$$Z = -\left[x\cos\phi_3 - \frac{R_1 - R}{\sin\phi_2}\right]\cos\phi_2 + \left[x\sin\phi_3\cos\theta\right]\sin\phi_2 \;.$$

B.3 <u>First Fundamental Form</u>

Following the procedure set forth for the sphere the radius vector in terms of x , θ is

$$\underset{\sim}{\bar{r}} = \left\{\left[x\cos\phi_3 - \frac{(R_1 - R)}{\sin\phi_2}\right]\sin\phi_2 + \left[x\sin\phi_3\cos\theta\right]\cos\phi_2\right\}i + \left(x\sin\phi_3\sin\theta\right)j$$
$$+\left\{-\left[x\cos\phi_3 - \frac{(R_1 - R)}{\sin\phi_2}\right]\cos\phi_2 + \left[x\sin\phi_3\cos\theta\right]\sin\phi_2\right\}k \;, \qquad (I-2-14)$$

and the derivative of $\underset{\sim}{\bar{r}}$ with respect to x and θ are

$$\underset{\sim}{\bar{r}}_{,x} = \left(\cos\phi_3\sin\phi_2 + \sin\phi_3\cos\phi_2\cos\theta\right)i + \left(\sin\phi_3\sin\theta\right)j$$
$$+\left(-\cos\phi_3\cos\phi_2 + \sin\phi_3\sin\phi_2\cos\theta\right)k \;, \qquad (I-2-15a-b)$$

$$\underset{\sim}{\bar{r}}_{,\theta} = -x\sin\phi_3\cos\phi_2\sin\theta\, i + x\sin\phi_3\cos\theta\, j - x\sin\phi_3\sin\phi_2\sin\theta\, k \;.$$

Then the Lamé parameters A_1 , A_2 are found to be

$$A_1 = 1 \quad ,$$
$$A_2 = x \sin\phi_3 \quad , \tag{I-2-16}$$

and since $\underset{\sim}{\bar{r}}_{,x} \cdot \underset{\sim}{\bar{r}}_{,\theta} = 0$, x , θ form an orthogonal system of curvilinear coordinates.

The unit base vectors of the middle-surface are

$$\underset{\sim}{\bar{a}}_1 = \left(\cos\phi_3 \sin\phi_2 + \sin\phi_3 \cos\phi_2 \cos\theta\right) i + \sin\phi_3 \sin\theta \, j$$
$$+ \left(-\cos\phi_2 \cos\phi_3 + \sin\phi_3 \sin\phi_2 \cos\theta\right) k \quad ,$$
$$\underset{\sim}{\bar{a}}_2 = -\sin\theta \cos\phi_2 \, i + \cos\theta \, j - \sin\phi_2 \sin\theta \, k \quad , \tag{I-2-17a-c}$$
$$\underset{\sim}{a}_3 = \left(-\sin\phi_3 \sin\phi_2 + \cos\phi_3 \cos\phi_2 \cos\theta\right) i + \cos\phi_3 \sin\theta \, j$$
$$+ \left(\sin\phi_3 \cos\phi_2 + \sin\phi_2 \cos\phi_3 \cos\theta\right) k \quad .$$

B.4 Second Fundamental Form

In a manner similiar to the sphere the principal radii of curvatures for the cone are determined by equation (I-2-9) and equation (I-2-17). After performing the required operations the principal radii of curvature are found to be

$$r_1 = \infty \quad ,$$
$$r_2 = x \tan\phi_3 \quad . \tag{I-2-18}$$

C. Geometry of Cylinder

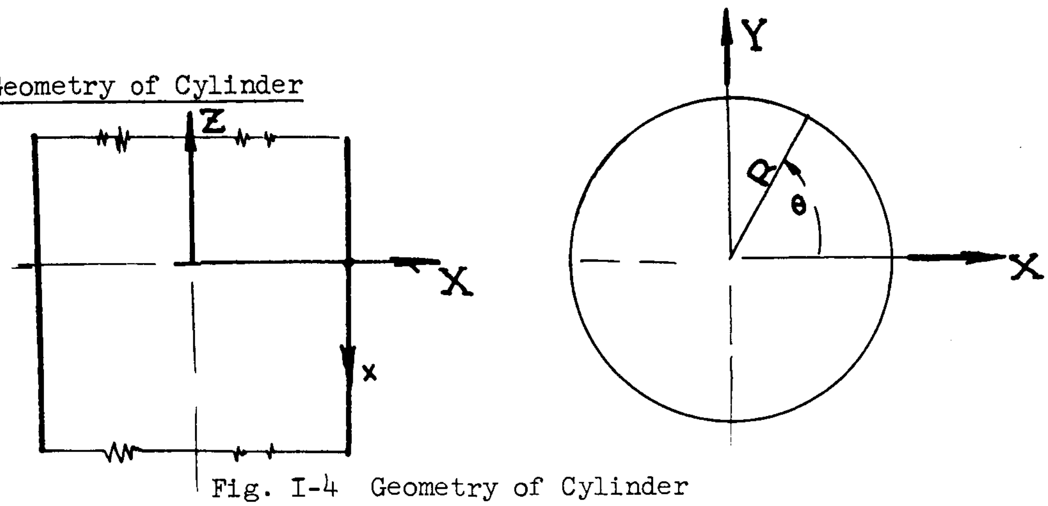


Fig. I-4 Geometry of Cylinder

C.1 Equation of Cylinder

The equation of a cylinder in the rectangular coordinate system X , Y , Z is

$$f_w = X^2 + Y^2 - R^2 = 0 \quad . \qquad \text{(I-2-19)}$$

C.2 Curvilinear Coordinates

If the curvilinear coordinates $X^1 = x$; $X^2 = \theta$ shown in Fig. I-4 are introduced, then the relation between X , Y , Z and x , θ is

$$X = R\cos\theta \ ,$$

$$Y = R\sin\theta \ , \qquad \text{(I-2-20a-c)}$$

$$Z = -x \quad .$$

C.3 First Fundamental form

The radius vector $\tilde{r}$ in terms of x , θ is

$$\tilde{\bar{r}} = R\cos\theta i + R\sin\theta j - xk \ , \qquad \text{(I-2-21)}$$

and the derivatives with respect to x and θ are

$$\underset{\sim}{\bar{r}}_{,x} = -k \quad , \qquad \text{(I-2-22)}$$

$$\underset{\sim}{\bar{r}}_{,\theta} = -R\sin\theta i + R\cos\theta j \quad .$$

The Lamé parameters A_1 and A_2 are found to be

$$A_1 = 1 \quad , \qquad \text{(I-2-23)}$$

$$A_2 = R \quad ,$$

and since $\underset{\sim}{\bar{r}}_{,x} \cdot \underset{\sim}{\bar{r}}_{,\theta} = 0$; x, θ form an orthogonal system of curvilinear coordinates.

The unit base vectors of the middle surface are

$$\underset{\sim}{\bar{a}}_1 = -k \quad , \qquad \text{(I-2-24a-c)}$$

$$\underset{\sim}{\bar{a}}_2 = -\sin\theta i + \cos\theta j \quad ,$$

$$\underset{\sim}{a}_3 = \cos\theta i + \sin\theta j \quad .$$

C.4 <u>Second Fundamental form</u>

In a manner similiar to the sphere the principal radii of curvature for the cylinder are determined by equation (I-2-9) and equation (I-2-24). After performing the required operations the principal radii of curvature are found to be

$$r_1 = \infty \quad , \qquad \text{(I-2-25a,b)}$$

$$r_2 = R \quad .$$

D. Geometry of Plates

D.1 Equation of Plates

The equation of the plate which intersects the cylinder, sphere and cone is given by

$$f_{P_1} = Y - \tan\phi_{P_1}\left[(R_1 - R) + X\right] = 0 \quad . \tag{I-2-26}$$

The equation of the plate which intersects the cone is given by

$$f_{P_2} = X + R_1 - R - R_o = 0 \quad . \tag{I-2-27}$$

1.3 Geometry of the Intersection of the Shell Elements

The intersection of two surfaces is a curve common to both. This curve has coordinates obtained by solving the equation of the two surfaces simultaneously. If $\underset{\sim}{\bar{r}}$ is the radius vector of the curve (Fig. I-5) then the unit vector tangent to the curve is

$$\underset{\sim}{\bar{a}}_\eta = \frac{d\underset{\sim}{\bar{r}}}{ds} \quad . \qquad \text{(I-3-1)}$$

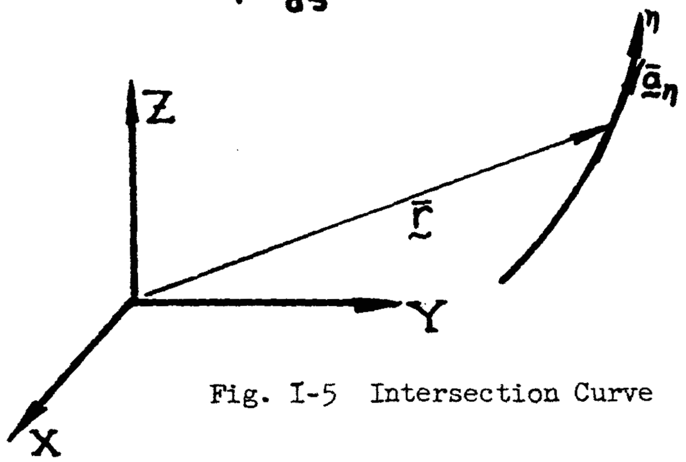


Fig. I-5 Intersection Curve

The geometry of the intersection is important in the consideration of the boundary conditions of the shell elements. It is shown in section 3.6 that the angle between $\underset{\sim}{\bar{a}}_\eta$ and $\underset{\sim}{\bar{a}}_1$ is necessary to define the boundary conditions. This angle is obtained by

$$\cos\lambda = \underset{\sim}{\bar{a}}_1 \cdot \underset{\sim}{\bar{a}}_\eta \quad . \qquad \text{(I-3-2)}$$

To determine $\cos\lambda$ an expression for $\underset{\sim}{\bar{a}}_\eta$ is needed. This is given by

$$\underset{\sim}{\bar{a}}_\eta = \frac{d\underset{\sim}{\bar{r}}}{ds} = \pm\frac{A_1\underset{\sim}{\bar{a}}_1\, d\alpha + A_2\underset{\sim}{\bar{a}}_2\, d\beta}{\left[A_1^2(d\alpha)^2 + A_2^2(d\beta)^2\right]^{\frac{1}{2}}} \,, \qquad \text{(I-3-3)}$$

when the coordinates coincide with orthogonal lines of curvature. For the intersection curve the coordinates are not independent and are related by the intersection equation. If the intersection equation is differentiated implicitly and the definition

$$d\beta = f\, d\alpha \tag{I-3-4}$$

is used, then equation (I-3-3) can be written as

$$\underset{\sim}{\bar{a}}_\eta = \pm \frac{A_1 \bar{a}_1 + A_2 \underset{\sim}{\bar{a}}_2 f}{\left[A_1^2 + A_2^2 f^2\right]^{\frac{1}{2}}} \; . \tag{I-3-5}$$

The direction cosine is now given by

$$\cos\lambda = \underset{\sim}{\bar{a}}_1 \cdot \underset{\sim}{\bar{a}}_\eta = \pm \frac{A_1}{\left[A_1^2 + A_2^2 f^2\right]^{\frac{1}{2}}} \; , \tag{I-3-6}$$

where f is defined by equation (I-3-4) and the intersection relation is imposed. Following is a summary of the intersection equations and direction cosines of the bulkhead shells.

A. Sphere

A.1 Sphere and Cylinder Intersection

The intersection of the sphere and cylinder is found by solving equation (I-2-1) and equation (I-2-18) simultaneously to yield

$$Z = 0 \; .$$

Then by setting equation (I-2-2c) equal to zero the required intersection equation between ϕ and θ can be obtained as

$$\cos\theta = -\cot\phi_2 \cot\phi \; . \tag{I-3-7}$$

The direction cosine for this intersection is obtained by equation (I-3-6) and equation (I-3-7) as

$$\cos\lambda = \pm \sin\phi_2 \sin\theta \; . \tag{I-3-8}$$

A.2 Sphere and Plate Intersection

An equation between φ and θ at the intersection of the sphere and plate is found by substituting equation (I-2-2a, b) into equation (I-2-26) to yield

$$\sin\theta - \tan\phi_1 \cos\phi_2 \cos\theta - \frac{\tan\phi_1}{\sin\phi}\left[\left(\frac{R_1}{R} - 1\right) - \sin\phi_2 \cos\phi\right] = 0. \quad \text{(I-3-9)}$$

Since equation (I-3-9) can not be solved explicitly for θ or φ a simple expression can not be obtained for the direction cosine. The equations required to solve for $\cos\lambda$ are:

$$\cos\lambda = \pm \frac{1}{\left[f^2 \sin^2\phi + 1\right]^{\frac{1}{2}}} ,$$

$$f = \frac{\tan\phi_1}{\sin\phi}\left[\frac{-\left(\frac{R_1}{R} - 1\right)\cos\phi + \sin\phi_2}{\cos\theta + \tan\phi_1 \cos\phi_2 \sin\theta}\right] , \quad \text{(I-3-10a-b)}$$

and equation (I-3-9).

A.3 Sphere and Cone Intersection

Referring to Fig. I-1, it is seen that the sphere intersects the cone such that

$$\phi = \phi_4 , \quad \text{(I-3-11a-b)}$$

$$0 \leq |\theta| \leq \theta_c ,$$

where θ_c is found by equation (I-3-9) in which ϕ is set equal to ϕ_4 .

The direction cosine along this intersection is

$$\cos\lambda = 0 .$$

B. Cone

B.1 Cone and Sphere Intersection

In terms of the coordinates of the cone the intersection of the cone and sphere is

$$X = X_c = \overline{bd} = (R_1 - R)\frac{\cos\phi}{\sin\phi} \quad , \qquad \text{(I-3-13a-b)}$$

$$0 \leq |\theta| \leq \theta_c \quad ,$$

where θ_c has been defined in equation (I-3-11).

Again the direction cosine is found to be

$$\cos\lambda = 0 \quad . \qquad \text{(I-3-14)}$$

B.2 Cone and Plate #1

If equation (I-2-13) is substituted into equation (I-2-26), the required intersection equation is found as

$$\sin\theta - \tan\phi_1 \cos\phi_2 \cos\theta = \frac{\tan\phi_1}{\sin\phi_3}\left[\cos\phi_3 \sin\phi_2\right] \quad .$$

It can be shown that the solution of this equation for θ yields $\theta = \theta_c$.

Thus the intersection relations are

$$0 \leq X \leq X_c \quad ,$$

$$\theta = \pm\theta_c \quad , \qquad \text{(I-3-15)}$$

It is found that $\cos\lambda$ along this cone plate intersection is given by

$$\cos\lambda = \pm 1 \quad . \qquad \text{(I-3-16)}$$

B.3 Cone and Plate #2

In a similiar manner as plate #1, the intersection of the cone and plate #2 is found by substituting equation (I-2-13a) into equation (I-2-27) to yield

$$X = \frac{R_0}{\left[\cos\phi_3 \sin\phi_2 + \sin\phi_3 \cos\phi_2 \cos\theta\right]} \quad , \tag{I-3-17}$$

for all θ . However, θ is restricted to $\pm\theta_c$ by plate #1.

Along this intersection

$$\cos\lambda = \frac{\pm \sin\theta \cos\phi_2}{\left[\cos^2\phi_2 \sin^2\theta + (\cos\phi_3 \sin\phi_2 + \sin\phi_3 \cos\phi_2 \cos\theta)^2\right]^{\frac{1}{2}}} \quad . \tag{I-3-18}$$

D. Cylinder

D.1 Cylinder and Sphere Intersection

As seen before the cylinder intersects the sphere at $Z = 0$. Thus, the intersection relations are

$$x = 0 \quad , \quad \text{for all } \theta \quad , \tag{I-3-19}$$

and the direction cosine is

$$\cos\lambda = 0 \quad . \tag{I-3-20}$$

D.2 Cylinder and Plate #1

From Fig. 1-1 it is seen that the cylinder intersects the plate such that

$$\theta = \pm\phi_6 \quad , \quad \text{for all } x \quad . \tag{I-3-21}$$

The direction cosine is

$$\cos \lambda = \pm 1 \quad . \qquad \text{(I-3-22)}$$

Appendix II

II ANISOTROPIC PLATES AND CYLINDERS

II.1 Introduction

It is sometimes desirable to construct shells having high bending stiffness to increase its buckling strength or to make it capable of carrying concentrated loads. This increased bending stiffness is achieved by sandwich construction and by integral stiffeners on one side of the shell. The important thing which the analyst does not want to lose sight of is the possibility of sacrificing transverse shear stiffness in trade for a gain in bending stiffness. This can happen in two ways; either as a consequence of the transverse shear stiffness being decreased and approaching the bending stiffness in magnitude or as a consequence of a very "soft" core. In either case the analysis becomes appreciably more complicated in nature and does not apply without modifications to include the effects of shear deformations. It is important that a substantial increase in bending stiffness can be obtained without an appreciable increase in weight in these cases. In this appendix, consideration is given to plates and cylinders of uniform thickness stiffened by ribs and stringers integrally attached to one side as shown in Fig. II-1. The ribs and stringers are assumed to be orthogonal to each other.

If the spacing between ribs and stringers is large it is necessary to analyze the structure as if it were composed of shell panels and stiffeners. However, when the spacing is small it is desirable to consider the limiting case which is an anisotropic shell. The only difference between an isotropic and anisotropic shell is the form of Hooke's law. Thus it is first necessary to obtain

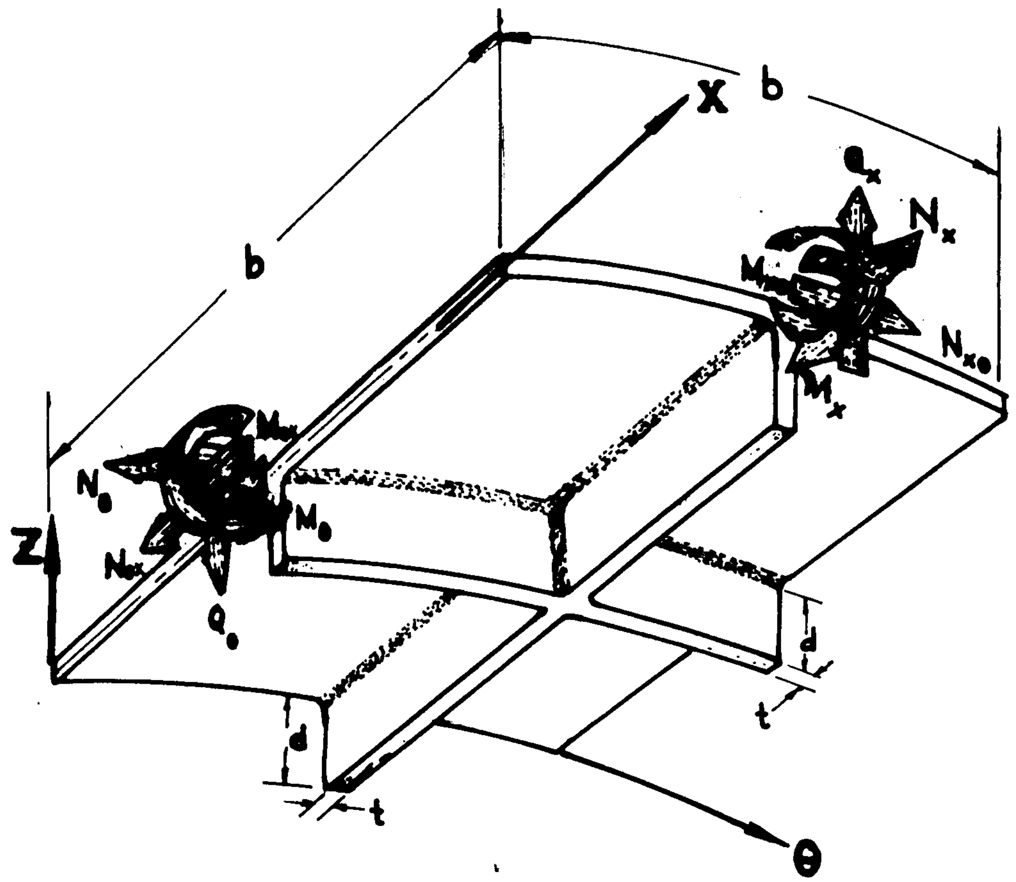


FIG II-1. ELEMENT OF STIFFENED SHELL

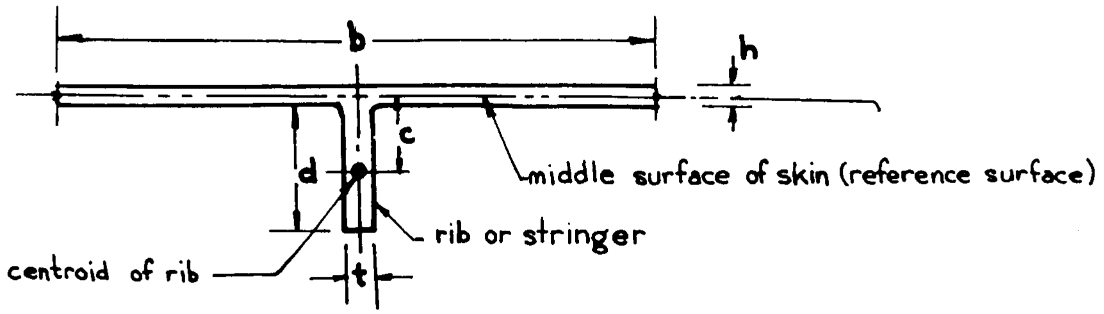


FIG. II-2. RIB AND STRINGER GEOMETRY

a set of constitutive equations, then the governing differential equations are obtained in a manner similar to the isotropic shell. The development follows that of Flügge[1] while the notation is that of Dow[2].

II.2 Constitutive Equations

In order to write the constitutive equations for a shell element, the stress-strain relations must be known for individual layers. If it is assumed that the ribs and stringers are orthogonal and are oriented in the direction of the principal coordinates, a special type of anisotropy is obtained, namely, orthotropic. For an orthotropic material the stress-strain relations are

$$\begin{aligned} \sigma_1 &= E_{11}\,\epsilon_1 + E_{12}\,\epsilon_2 , \\ \sigma_2 &= E_{12}\,\epsilon_1 + E_{22}\,\epsilon_2 , \\ \tau_{12} &= G\,\gamma_{12} , \end{aligned} \qquad \text{(II-2-1a-c)}$$

where the strains are defined in terms of displacements by equation (3-4-12). To determine the stress resultants and couples for a shell element in which the ribs and stringers have the same dimensions and spacing [Fig. II-2] it is necessary to perform the operations

$$N_1 = \int_s \sigma_1\, dz + \int_r \sigma_1 \frac{t}{b}\, dz \quad , \qquad M_1 = \int_s \sigma_1\, z dz + \int_r \sigma_1 \frac{t}{b}\, dz ,$$

$$N_2 = \int_s \sigma_2\, dz + \int_r \sigma_2 \frac{t}{b}\, dz \quad , \qquad M_2 = \int_s \sigma_2\, z dz + \int_r \sigma_2 \frac{t}{b}\, dz ,$$

(II-2-2a-g)

[1] Flügge, W., Stresses in Shells, Springer-Verlag, Berlin, 1960

[2] Dow, N. F., Libove, C., and Hubka, R. E., Formulas for the Elastic Constants of Plates with Integral Waffle-Like Stiffening, NACA Report 1195, Washington D. C., 1954

$$N_{12} = \int_s \tau_{12}(1 + \frac{z}{r_2})\, dz \quad , \quad M_{12} = M_{12} = \int_s \tau_{12}\, zdz + \int_r \tau_{12} \frac{t}{b}\, zdz$$

$$N_{21} = \int_s \tau_{12}(1 + \frac{z}{r_1})\, dz \, ,$$

where s denotes integration through the skin and r the integration through the rib or stringer with the middle surface of the skin as reference. It is assumed that ribs and stringers strain in the direction of the applied load and bending of the stiffeners in the plane of the skin is small (i.e., in plane shear carried by skin). Then the constitutive equations for a plate and cylinder are

Plate

$$
\begin{aligned}
N_x &= E_x \bar{\epsilon}_x + E_\nu \bar{\epsilon}_y - S_c \chi_x \, , \\
N_y &= E_\nu \bar{\epsilon}_x + E_y \bar{\epsilon}_y - S_c \chi_y \, , \\
N_{xy} &= N_{yx} = E_{xy} \bar{\gamma} \, , \\
M_x &= D_x \chi_x + D_\nu \chi_y - S_c \bar{\epsilon}_x \, , \\
M_y &= D_\nu \chi_x + D_y \chi_y - S_c \bar{\epsilon}_y \, , \\
M_{xy} &= M_{yx} = D_{xy} \chi_{xy} \, .
\end{aligned}
\qquad \text{(II-2-3a-f)}
$$

Cylinder

$$
\begin{aligned}
N_x &= E_x \bar{\epsilon}_x + E_\nu \bar{\epsilon}_\theta - S_c \chi_x \, , \\
N_\theta &= E_\nu \bar{\epsilon}_x + E_\theta \bar{\epsilon}_\theta - S_c \chi_\theta \, , \\
N_{\theta x} &= E_{\theta x} \bar{\gamma} \, ,
\end{aligned}
$$

$$N_{x\theta} = E_{\theta x}\,\bar{\gamma} + \frac{D_{\theta x}}{R}\,\chi_{x\theta} \quad , \qquad \text{(II-2-4a-g)}$$

$$M_x = D_x\chi_x + D_\nu\chi_\theta - S_c\bar{\epsilon}_x \quad ,$$

$$M_\theta = D_\nu\chi_x + D_\theta\chi_\theta - S_c\bar{\epsilon}_\theta \quad ,$$

$$M_{x\theta} = M_{\theta x} = D_{\theta x}\chi_{x\theta} \quad ,$$

where

$$E_x = E_y = E_\theta = \frac{Eh}{1-\nu^2}\left[1 + \frac{t}{h}\frac{d}{b}(1-\nu^2)\right] \quad ,$$

$$E_\nu = \frac{\nu Eh}{1-\nu^2} \quad ,$$

$$E_{xy} = E_{\theta x} = \frac{Eh}{2(1+\nu)} \quad ,$$

$$S_c = CE\,t\left(\frac{d}{b}\right) \quad ,$$

$$D_x = D_y = D_\theta = \frac{Eh^3}{12(1-\nu^2)}\left\{1 + \frac{t}{h}\frac{d}{b}\left[\left(\frac{d}{h}\right)^2 + 12\left(\frac{c}{h}\right)^2\right](1-\nu^2)\right\} \quad ,$$

$$D_{xy} = D_{\theta x} = \frac{Eh^3}{12(1+\nu)}\left[1 \qquad 2\left(\frac{t}{h}\right)^3\frac{d}{b}\right], \quad D_\nu = \nu\frac{Eh^3}{12(1-\nu^2)} \quad .$$

If the stiffeners are not oriented in the coordinate directions new stiffness coefficients can be obtained by a rotation of the stiffeners. When a rotation of 45° is performed the constitutive equations for a plate and cylinder become

<u>Plate</u>

$$N_x = E_1\bar{\epsilon}_x + E_2\bar{\epsilon}_y - \frac{1}{2}S_c\left[\chi_x + \chi_y\right] \quad ,$$

$$N_y = E_2\bar{\epsilon}_x + E_1\bar{\epsilon}_y - \frac{1}{2}S_c\left[\chi_x + \chi_y\right] \quad ,$$

$$N_{xy} = N_{yx} = \left[E_{xy} + \frac{1}{2}\frac{S_c}{C}\right]\bar{\gamma} - \frac{1}{2}S_c\chi_{xy} \quad , \qquad \text{(II-2-5a-f)}$$

$$M_x = D_1\chi_x + D_2\chi_y - \frac{1}{2}S_c\left[\bar{\epsilon}_x + \bar{\epsilon}_y\right] \quad ,$$

$$M_y = D_2\chi_x + D_1\chi_y - \frac{1}{2}S_c\left[\bar{\epsilon}_x + \bar{\epsilon}_y\right] \quad ,$$

$$M_{xy} = M_{yx} = \frac{1}{2} D_{12} \chi_{xy} - \frac{1}{2} S_c \bar{\gamma}$$

Cylinder

$$N_x = E_1 \bar{\epsilon}_x + E_2 \bar{\epsilon}_\theta + \left[\frac{D_{x\theta}}{4R} - \frac{S_c}{2}\right] \chi_x - \left[\frac{D_{x\theta}}{4R} + \frac{S_c}{2}\right] \chi_\theta ,$$

$$N_\theta = E_2 \bar{\epsilon}_x + E_1 \bar{\epsilon}_\theta - \left[\frac{D_{x\theta}}{4R} + \frac{S_c}{2}\right] \chi_x + \left[\frac{D_{x\theta}}{4R} - \frac{S_c}{2}\right] \chi_\theta ,$$

$$N_{\theta x} = \left[E_{x\theta} + \frac{1}{2} \frac{S_c}{C}\right] \bar{\gamma} - \frac{1}{2} S_c \chi_{x\theta} - \frac{D_{x\theta}}{4R} \left[\chi_\theta - \chi_x\right] ,$$

$$N_{x\theta} = \left[E_{x\theta} + \frac{1}{2} \frac{S_c}{C}\right] \bar{\gamma} - \frac{1}{2} S_c \chi_{x\theta} + \frac{D_{x\theta}}{4R} \left[\chi_\theta - \chi_x\right] , \qquad \text{(II-2-6a-g)}$$

$$M_x = D_1 \chi_x + D_2 \chi_\theta - \frac{1}{2} S_c \left[\bar{\epsilon}_x + \bar{\epsilon}_\theta\right] ,$$

$$M_\theta = D_2 \chi_x + D_1 \chi_\theta - \frac{1}{2} S_c \left[\bar{\epsilon}_x + \bar{\epsilon}_\theta\right] ,$$

$$M_{x\theta} = M_{\theta x} = \frac{1}{2} D_{12} \chi_{x\theta} - \frac{1}{2} S_c \bar{\gamma} ,$$

where

$$E_1 = \frac{1}{2}\left[E_x + E_\nu + 2 E_{xy}\right] ,$$

$$E_2 = \frac{1}{2}\left[E_x + E_\nu - 2 E_{xy}\right] ,$$

$$D_1 = \frac{1}{2}\left[D_x + D_\nu + D_{xy}\right] ,$$

$$D_2 = \frac{1}{2}\left[D_x + D_\nu - D_{xy}\right] ,$$

$$D_{12} = \left[D_x - D_\nu\right] .$$

II.3 Governing Differential Equation

The governing differential equations for anisotropic shells in terms of u , v , w are obtained by the same procedure as set forth in section 3.4. The equations for plates and cylinders with ribs in the coordinate directions and rotated 45° off of the coordinate directions are as follows

a) Plate — Stiffeners in Coordinate Directions

$$E_x u,_{xx} + E_{xy}\, u,_{yy} + (E_{xy} + E_\nu)\, v,_{xy} + S_c w,_{xxx} = - p_x \quad ,$$

$$(E_{xy} + E_\nu)\, u,_{xy} + E_y v,_{yy} + E_{xy} v,_{xx} + S_c w,_{yyy} = - p_y \quad ,$$

$$S_c u,_{xxx} + S_c v,_{yyy} + D_x w,_{xxxx} + 2(D_\nu + D_{xy})\, w,_{xxyy} + D_y w,_{yyyy} = p_z \quad . \qquad \text{(II-3-1a-c)}$$

b) Cylinder — Stiffeners in Coordinate Directions

$$RE_x u,_{xx} + \frac{E_{\theta x}}{R}\, u,_{\theta\theta} + (E_\nu + E_{\theta x})\, v,_{x\theta} + RS_c w,_{xxx} + E_\nu w,_x = - Rp_x \quad ,$$

$$(E_{\theta x} + E_\nu)\, u,_{x\theta} + R\left(E_{\theta x} + 2\,\frac{D_{\theta x}}{R^2}\right) v,_{xx} + \frac{1}{R}\left(E_\theta + \frac{D_\theta}{R^2} - 2\,\frac{S_c}{R}\right) v,_{\theta\theta}$$
$$- \frac{1}{R}\,(D_\nu + 2D_{\theta x})\, w,_{xx\theta} - \frac{1}{R^2}\left(\frac{D_\theta}{R} - S_c\right) w,_{\theta\theta\theta} + \frac{1}{R}\left(E_\theta - \frac{S_c}{R}\right) w,_\theta = - Rp_\theta \quad , \qquad \text{(II-3-2a-c)}$$

$$- S_c u,_{xxx} - \frac{E_\nu}{R}\, u,_x + \frac{1}{R^2}\,(D_\nu + 2D_{\theta x})\, v,_{xx\theta} + \frac{1}{R^4}\,(D_\theta - RS_c)\, v,_{\theta\theta\theta}$$
$$+ \frac{1}{R^3}\,(S_c - RE_\theta)\, v,_\theta - D_x w,_{xxxx} - \frac{2}{R^2}\,(D_\nu + D_{\theta x})\, w,_{xx\theta\theta}$$
$$- \frac{D_\theta}{R^4}\, w,_{\theta\theta\theta} - 2\,\frac{S_c}{R^3}\, w,_{\theta\theta} - \frac{E_\theta}{R^2}\, w = -p_z \quad .$$

c) Plate — Stiffeners 45° from Coordinate Directions

$$E_1\, u,_{xx} + \left(E_{xy} + \frac{1}{2}\frac{S_c}{C}\right) u,_{yy} + \left(E_2 + E_{xy} + \frac{1}{2}\frac{S_c}{C}\right) v,_{xy} + \frac{1}{2} S_c\, w,_{xxx} + S_c\, w,_{xyy} = -\, p_x \ ,$$

$$\left(E_2 + E_{xy} + \frac{1}{2}\frac{S_c}{C}\right) u,_{xy} + \left(E_{xy} + \frac{1}{2}\frac{S_c}{C}\right) v,_{xx} + E_1\, v,_{yy} + S_c\, w,_{xxy} + \frac{1}{2} S_c\, w,_{yyy} = -\, p_y \ , \qquad \text{(II-3-3a-c)}$$

$$\frac{1}{2} S_c\, u,_{xxx} + \frac{3}{2} S_c\, u,_{xyy} + \frac{3}{2} S_c\, v,_{xxy} + \frac{1}{2} S_c\, v,_{yyy} + D_1\, w,_{xxxx} + \left[2D_2 + D_{12}\right] w,_{xxyy} + D_1\, w,_{yyyy} = p_z \ .$$

d) Cylinder — Stiffeners 45° from Coordinate Directions

$$\left(RE_1\right) u,_{xx} + \frac{1}{R}\left(E_{x\theta} + \frac{1}{2}\frac{S_c}{C}\right) u,_{\theta\theta} + \left[E_2 + E_{x\theta} - \frac{D_{x\theta}}{4R^2} + \left(\frac{1}{2C} - \frac{1}{R}\right) S_c\right] v,_{x\theta}$$
$$- \frac{D_{x\theta}}{4R^2}\, v,_{\theta\theta} - R\left(\frac{D_{x\theta}}{4R} - \frac{S_c}{2}\right) w,_{xxx} - \frac{D_{x\theta}}{4R}\, w,_{xx\theta}$$
$$+ \frac{1}{R}\left(\frac{D_{x\theta}}{4R} + S_c\right) w,_{x\theta\theta} + \frac{D_{x\theta}}{4R^3}\, w,_{\theta\theta\theta} + \frac{E_2}{R}\, w,_{x} = -\, Rp_x \ ,$$

$$\left[E_2 + E_{x\theta} + \left(\frac{1}{2C} - \frac{1}{R}\right) S_c\right] u,_{x\theta} + R\left[E_{x\theta} + \frac{D_{12}}{2R^2} + \left(\frac{1}{2C} - \frac{1}{R}\right) S_c\right] v,_{xx}$$
$$+ \frac{D_{x\theta}}{4R^2}\, v,_{x\theta} + \frac{1}{R}\left[E_1 + \frac{D_2}{R^2} + \frac{D_{x\theta}}{4R^2} - \frac{S_c}{R}\right] v,_{\theta\theta}$$
$$+ \frac{D_{x\theta}}{4}\, w,_{xxx} - \frac{1}{R}\left[D_2 - \frac{D_{x\theta}}{4} + \frac{D_{12}}{2} - RS_c\right] w,_{xx\theta} \qquad \text{(II-3-4a-c)}$$
$$- \frac{D_{x\theta}}{4R^2}\, w,_{x\theta\theta} - \frac{1}{R^3}\left[D_2 + \frac{D_{x\theta}}{4} - \frac{RS_c}{2}\right] w,_{\theta\theta\theta}$$
$$+ \frac{1}{R}\left[E_1 - \frac{1}{2}\frac{S_c}{R}\right] w,_{\theta} = -\, Rp_{\theta} \ ,$$

$$
\begin{aligned}
&\frac{S_c}{2} u,_{xxx} + \frac{3}{2R^2} u,_{x\theta\theta} + \frac{E_2}{R} u,_x + \left(\frac{S_c}{R} - \frac{D_2}{R^2} - \frac{D_{12}}{R}\right) v,_{xx\theta} \\
&\quad + \left(\frac{S_c}{2R} - \frac{D_1}{R^4}\right) v,_{\theta\theta\theta} + \left(\frac{E_1}{R^2} + \frac{D_{x\theta}}{4R^4} - \frac{S_c}{2R^3}\right) v,_\theta \\
&\quad + D_1 w,_{xxxx} + \left(2 \frac{D_2}{R^2} + \frac{D_{12}}{R}\right) w,_{xx\theta\theta} + \frac{D_1}{R^2} w,_{\theta\theta\theta\theta} \\
&\quad + \left(\frac{D_{x\theta}}{4R^2} + \frac{S_c}{R}\right) w,_{xx} + \left(\frac{S_c}{R^3} - \frac{D_{x\theta}}{4R^4}\right) w,_{\theta\theta} + \frac{E_1}{R^2} w = p_z \; .
\end{aligned}
$$

Appendix III

III GENERAL ASPECTS OF RESIDUAL STRESSES IN WELDED STRUCTURES

III.1 Introduction

Residual, inherent, locked-in or reaction stresses are the self-equilibrated stresses within a structure when the surface and inertia loads are absent. These stresses are caused primarily by the initiation of localized plastic deformations of a body. Although plastic deformations may be induced by various means such as welding, forming, non-uniform cooling, cold-working, restraint of deformations and phase transformations of the materials, etc., as far as the structural integrity is concerned, it is believed that those initiated by welding under restraint are the most serious ones to cope with.

The significance of the residual stresses in cold wrought metals was observed about half a century ago by E. Heyn. The effect of these locked-in stresses on the strength of structures and ships, however, was recognized only within the last two decades because of the large-scale brittle failures of the all-welded ships and bridges experienced during the period of World War II. In spite of the fact that both theoretical and experimental studies have been carried out by various investigators, facts concerning the causes of failures of such structures and their prevention are still not thoroughly understood. The obvious reason for this is mainly attributed to the complexity involved in the failure mechanism. As a result of these investigations, however, the essential phenomena regarding the role of residual stresses are revealed to a certain extent, and different methods of evaluating such stresses have been developed. The purpose of the present work is to present an up-to-date and brief resume of the general effects of residual stresses in welded structures.

III.2 Methods of Evaluation

Method of investigating the magnitude and distribution of residual stresses may be classified as (A) Analytical and (B) Experimental. The latter method may be further subclassified as (B-1) Destructive, (B-2) Semi-destructive and (B-3) Non-destructive.

A. Analytical Method. If the time history of temperature distribution within the structure due to welding and stress-strain relation of the materials at various temperatures are known, residual stresses may be obtained from an elastic-plastic analysis, using the flow theory of plasticity and appropriate yield condition. Since this is a nonlinear problem and the numerical analysis can only be performed stepwise, choosing small-time increments for the individual solutions, and results are then accumulated successively. FigureIII.1 shows schematically how the residual stress in the middle portion of the center-welded plate is generated. It should be noted that this mathematical analysis is rather complicated and it is not exact because accurate information of the temperature distribution and its variation with time is very difficult to obtain, especially for multi-pass welding processes.

B. Experimental Methods

B.1 Destructive Methods

B.1.1 Method of Sectioning. This method is applied to welded structures composed of thin plate elements in which the locked-in stresses are essentially uniform through the thickness. The basic assumption used is that the residual stresses are elastic. If the plates are cut, the relieved strains can be measured by means of suitable gages and then the stresses are computed from the linear stress-strain relations. The technical procedure involves (a) installing the strain gages on the surface of the plates after the welding process, (b) calibrating the gages, (c) sectioning of the plates, (d) reading the relaxed strains and (e) computing the stresses from the elastic law.

B.1.2 Method of Trepanning or Boring. This method is used to obtain both surface and internal residual stresses of thick plates and tubes. The principle involved is essentially the same as that of sectioning except that the technique is refined such that the cutting through the thickness is performed in a sequential manner and intermediate strains are recorded after each layer is trepanned or bored. Consequently, stresses at different locations across the thickness are evaluated.

B.2 Semi-Destructive Method

B.2.1 Method of the Hole. This method involves the application of the known stress-strain relations for thin plates with and without a hole. The residual stresses are obtained through the differential strains measured experimentally before and after a hole is cut. Since the size of hole to be cut is small (about 1/2" diameter), the evaluated stresses are quite accurate. It should be remarked that strain rosettes were exclusively placed outside of the hole in the previous works. It is expected that better results, perhaps, may be obtained for the future studies if they are cemented within the hole since improved smaller strain gages have been developed lately.

B.3 Non-Destructive Methods

B.3.1 Photoelastic Method. This method has been used to study the quantitative residual stress distribution in welded plates by simulating a weld in the transparent materials such as the photoelastic bakelite. The pattern of the frozen or residual stresses can be observed through the polariscope using polarized light. The fringe values are then converted into stresses from the stress-optic law.

B.3.2 X-Ray Diffraction Method. The residual stresses in the regions away from the weld can be evaluated by this technique. It involves the determination of the locked-in strains which are assumed to be proportional to the changes in the interactomic distances between planes of atoms. This can be done by measuring the Bragg angle from 2-exposure x-ray diffractions, and the corresponding stresses are expressible in terms of the material properties and change of Bragg angle.

It is believed that this is the only method to obtain the quantitative residual stresses nondestructively. This method also provides a better way of measuring steep gradients because it can determine strains within a very small area. The disadvantages of this method are: (a) only the surface stresses on smooth surfaces can be observed, (b) the grain size of the material must be neither too large nor too small, and (e) the stresses in the weld are difficult to determine.

III.3 Stress or Strain Patterns

Some stress or strain patterns of some simple (as-delivered, as-welded and heat-treated) structures are shown below. The strains are determined by the destructive methods.

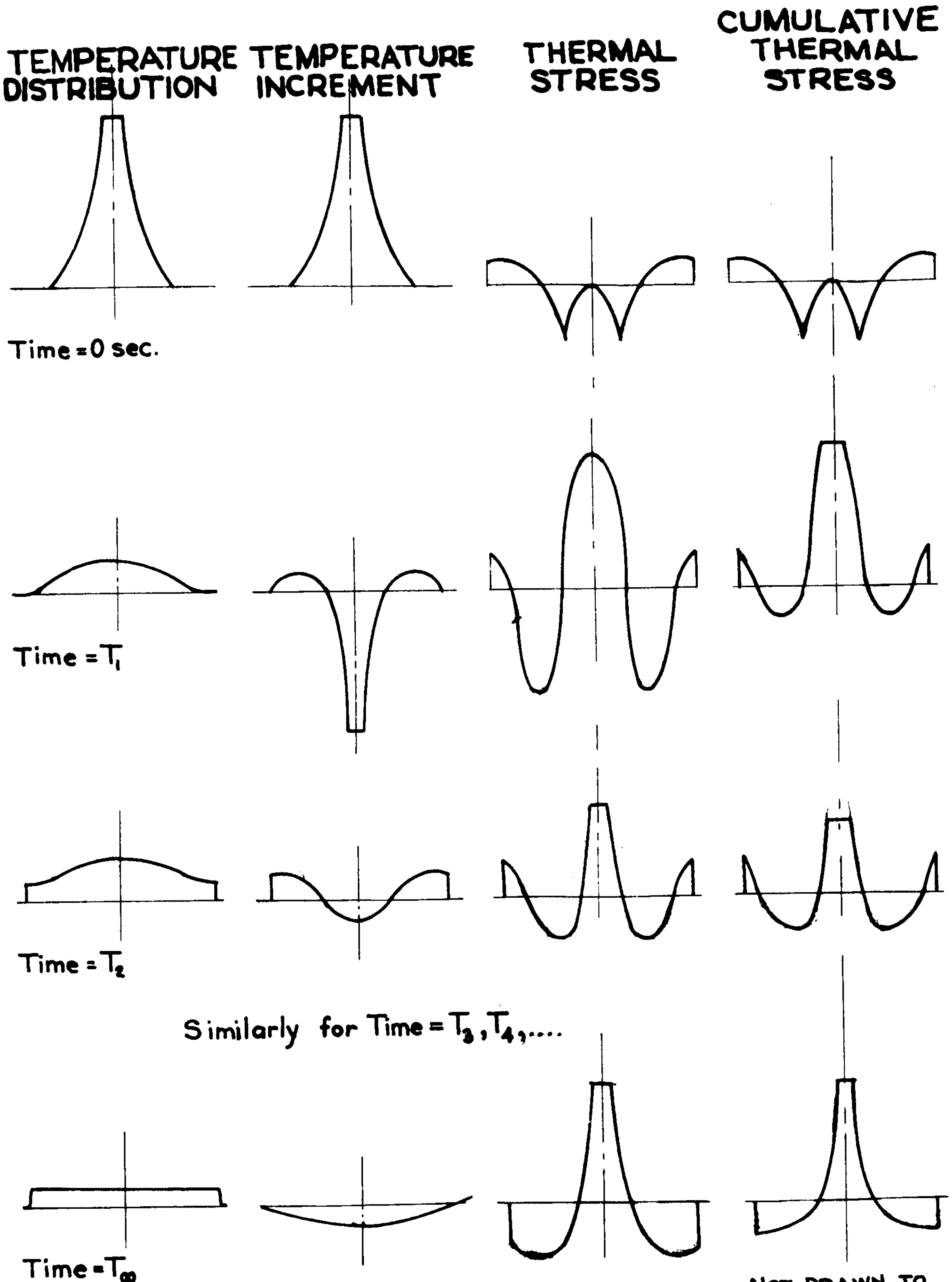


FIG. III-1
FORMATION OF RESIDUAL STRESSES IN A CENTER-WELDED PLATE

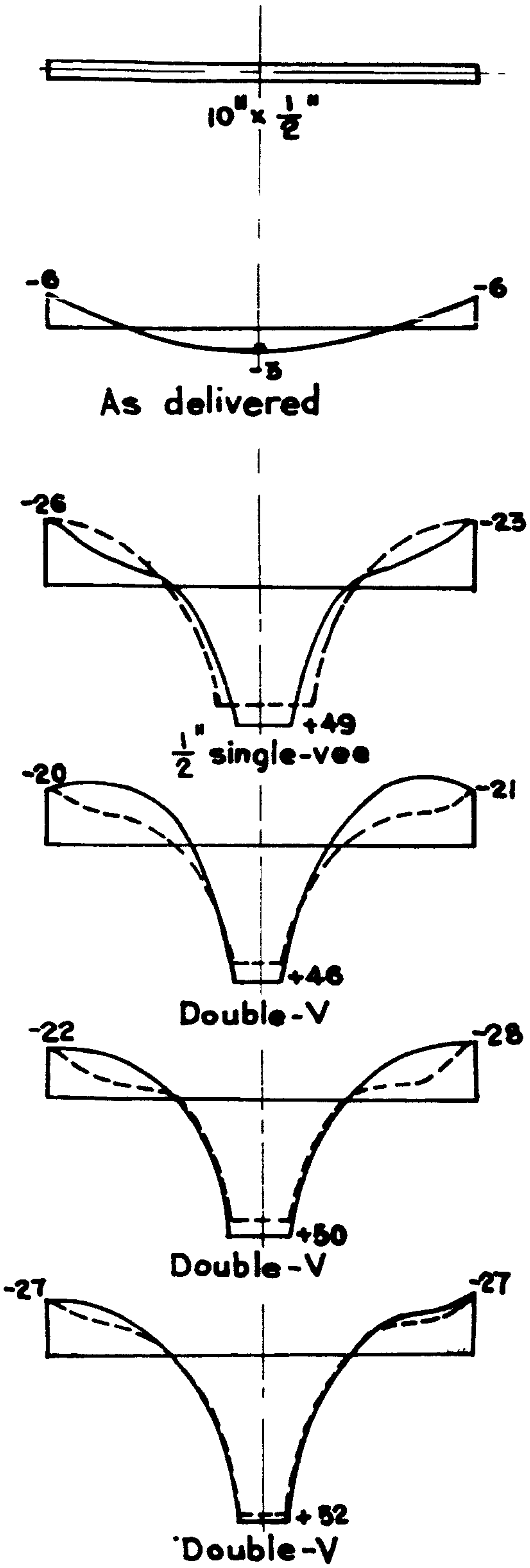


FIG. III-2 RESIDUAL STRESS DISTRIBUTION, CENTER-WELDED PLATE

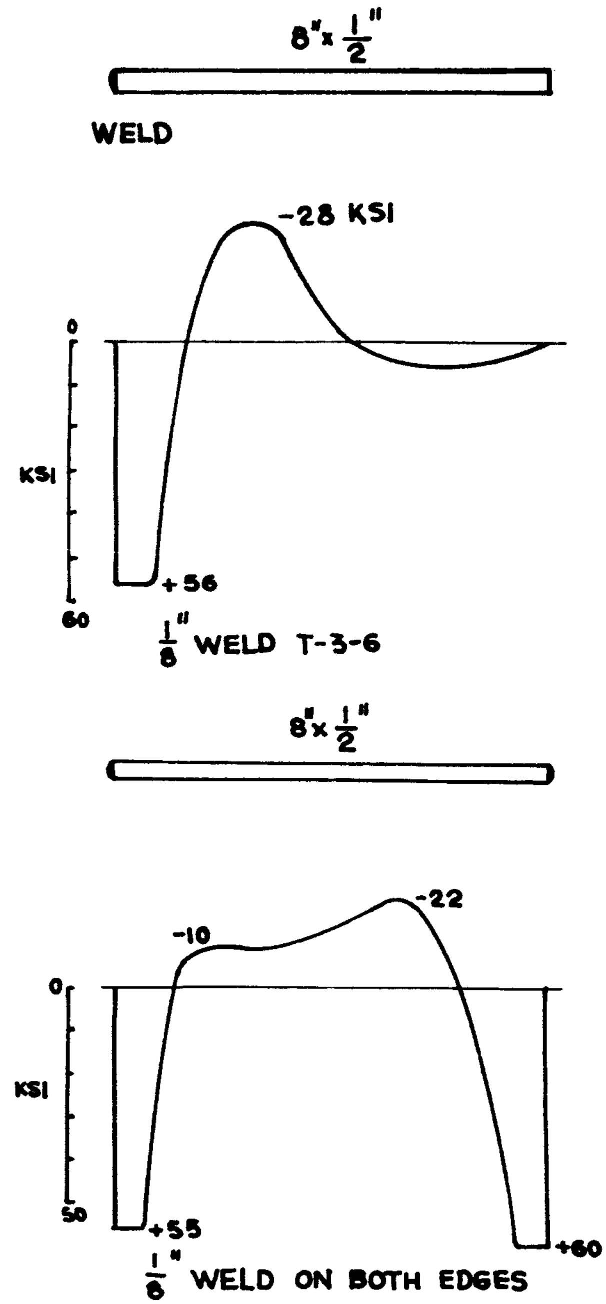


FIG. III-3 RESIDUAL STRESS DISTRIBUTION, EDGE-WELDED PLATE

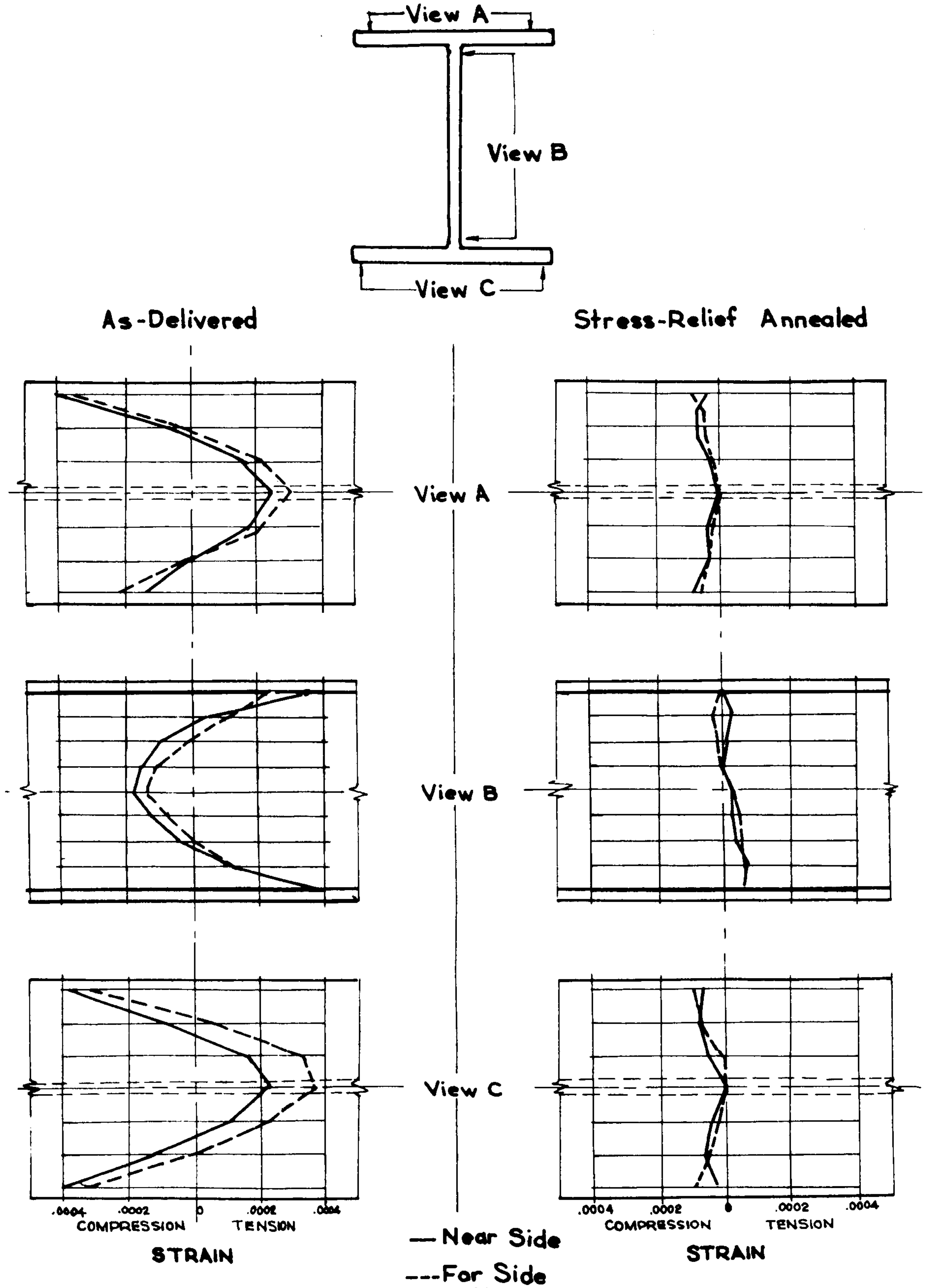


FIG III-4 RESIDUAL STRAINS FORMED DUE TO COOLING AFTER ROLLING 8WF67 BEAM

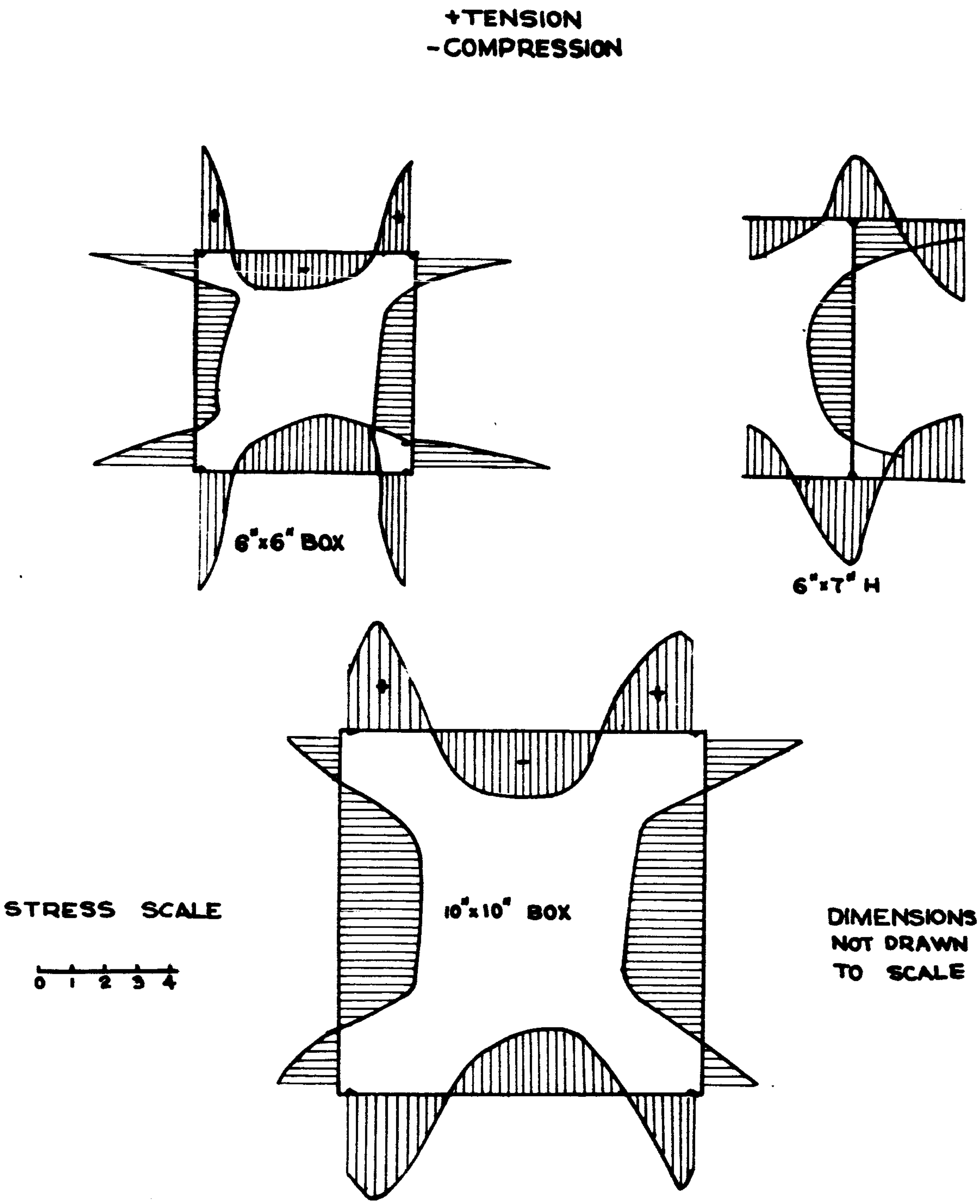
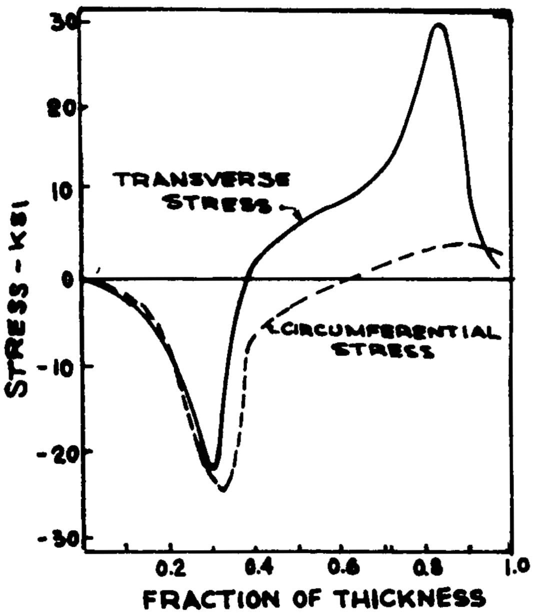
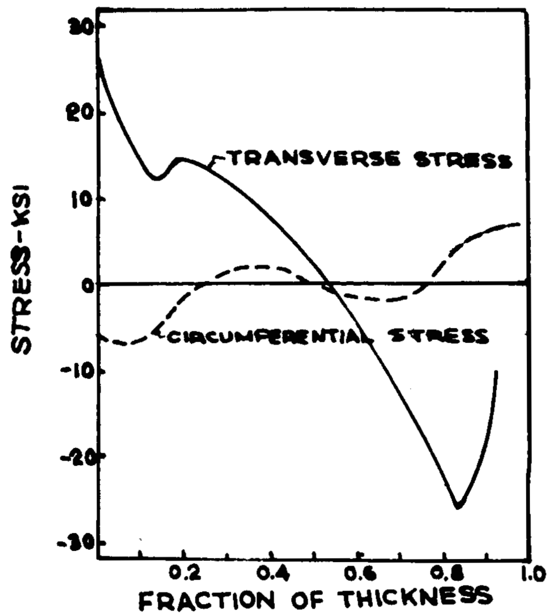


FIG. III-5
TYPICAL RESIDUAL STRESS DISTRIBUTIONS

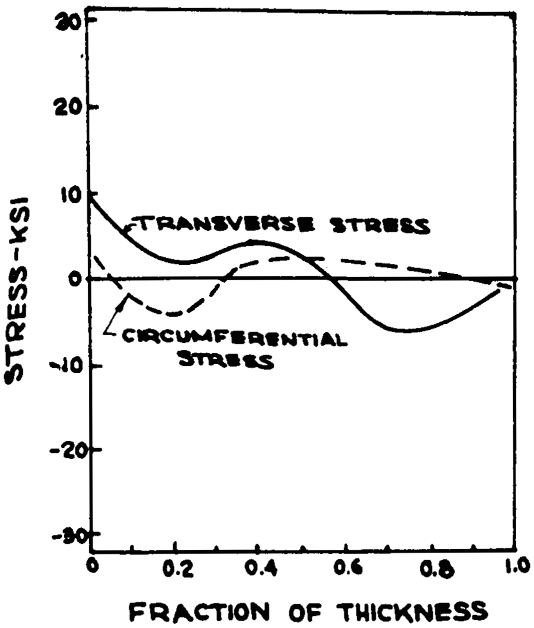
FIGURES III 6(a)-6(d)



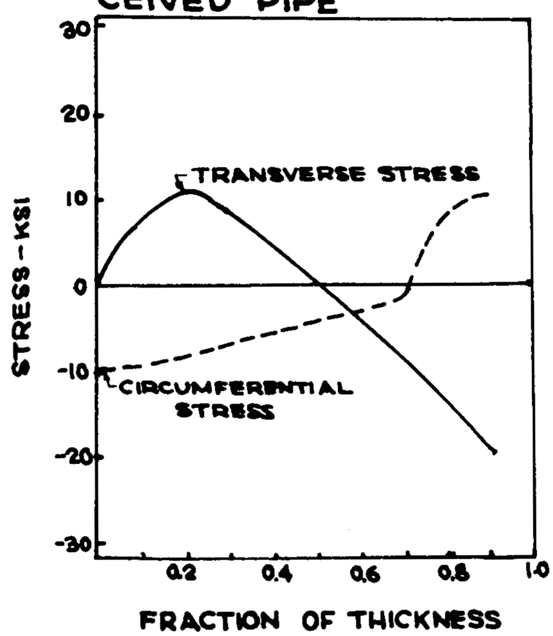
(a) AVERAGE STRESS DISTRIBUTIONS IN AS RECEIVED PIPE



(b) AVERAGE STRESS DISTRIBUTIOINS IN WELDS OF AS-WELDED, AS RECEIVED PIPE



(c) AVERAGE STRESS DISTRIBUTIONS IN WELDS MADE WITH 400°F PREHEAT AND 1200°F STRESS RELIEF POSTHEAT



(d) AVERAGE STRESS DISTRIBUTIONS IN WELD MADE IN ANNEALED PIPE

III.3.1 Stress Distribution in Plates

Fig. III-2 shows the typical longitudinal stress pattern of some as-delivered and various center-welded plates. Fig. III-3 shows the corresponding stress distribution of the edge-welded plates. These plates are of ASTM Designation A7 Structural Steel, welded manually.

III.3.2 Strain and Stress Distributions of WF and Box Members

Fig. III-4 shows the strain distributions for both as-delivered and stress-relief annealed cases of a non-welded WF beam. Fig. III-5 shows the general stress distributions of welded H and box sections. All base materials are of A7 Steel. The joints were machine welded using automatic submerged-arc.

III.3.3 Variations of Stresses in Welded Pipe

Fig. III-6 shows the stress variations due to circumferential single-V butt welds in 5-1/2 inch mild-steel pipe, 1/2-inch thick, with different welding procedures.

III.4 Discussion and Recommendations

In view of the stress patterns as shown in the foregoing section and the associated results given by various investigators (see references listed at the end of this appendix), the following general conclusion can be drawn:

a) Residual stresses are usually present in the unwelded wrought product.

b) Residual stresses in welded structures vary with the structural configuration, the weld design, the welding process, pre- and post-weld history, the stress relaxation characteristics of base metal, weld assemblage procedures, etc.

c) Residual peak stresses in weldments which are not stress relieved are governed by the stress relaxation characteristics of the base metal. An annealing treatment would remove residual welding stresses completely, however, any measured residual stress after annealing would be the result of differential cooling from annealing temperatures. Artificial aging after welding as applied in aluminum alloys would also reduce residual stresses.

d) Bending strength is reduced when an external load is superimposed on a residual tension stress field. Residual stresses may influence the buckling characteristics of a column, however, a weldment is a metallurgical discontinuity and its mere presence could contribute to lowering buckling strengths significantly, especially in columns of low or medium slenderness ratios.

e) The magnitude of transverse stresses perpendicular to the weld in plate structures is usually small as compared to those along the weld. However, the pattern of transverse stress distribution has not given as much attention as that for longitudinal stresses.

It should be noted that previous studies on residual stresses were limited in simple steel structures, and very little work can be found on welded structures made of aluminum alloys and shapes involving a change in thickness with various end restraints, using different types of welding procedures. It is expected that new stress patterns may be found, especially in the vicinity of welds with restraints such as joining a Y-extrusion with tapered plates and ribbed shell elements in multicellular pressure vessels. Furthermore, when such all-welded large structures are not stress relieved, it is expected the two-dimensional residual stresses are very high, and the longitudinal stresses within the weld may eventually reach the yield limit. If one wants to find the joint efficiency by means of the ordinary uni-axial standard coupon testing procedure, unreasonable results may be produced due primarily to neglecting of the simultaneous action of the longitudinal stresses, unless large specimens are tested bi-axially. The procedure of the latter method of testing is rather involved and difficult to perform, however.

Residual stresses are a special kind of pre-stress, and they can be incorporated into the theoretical analysis by simple superposition or by establishing a reasonable joint efficiency of weld. Both methods require that the magnitude and distribution of stresses are known. The joint efficiency is established through a well-known yield criterion of failure using energy approach and uni-axial test results. Unfortunately, the analytical method of evaluating the residual stresses, as mentioned before, is very difficult and experimental methods are less complicated. Consequently, it is recommended that some experimental work on the accurate determination of the residual stresses should be carried out. Investigations should also be conducted on the effects of various possible means of stress-relief of residual stresses.

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Appendix IV

IV. A DIRECT METHOD OF SOLVING LARGE MATRICES OF FINITE-DIFFERENCE EQUATIONS

IV.1 Introduction

The generation of a set of simultaneous finite difference equations to replace the original governing differential equations has been discussed in section 6.2. This appendix contains a description of a method used to solve the finite-difference system and a discussion of a digital computer program for a cylindrical shell segment. The same general method is applicable with slight modifications to other geometries.

A typical rectangular mesh for a cylinder whose boundaries coincide with mesh lines is shown in Fig. IV-1. Since there are three unknowns at each mesh point,

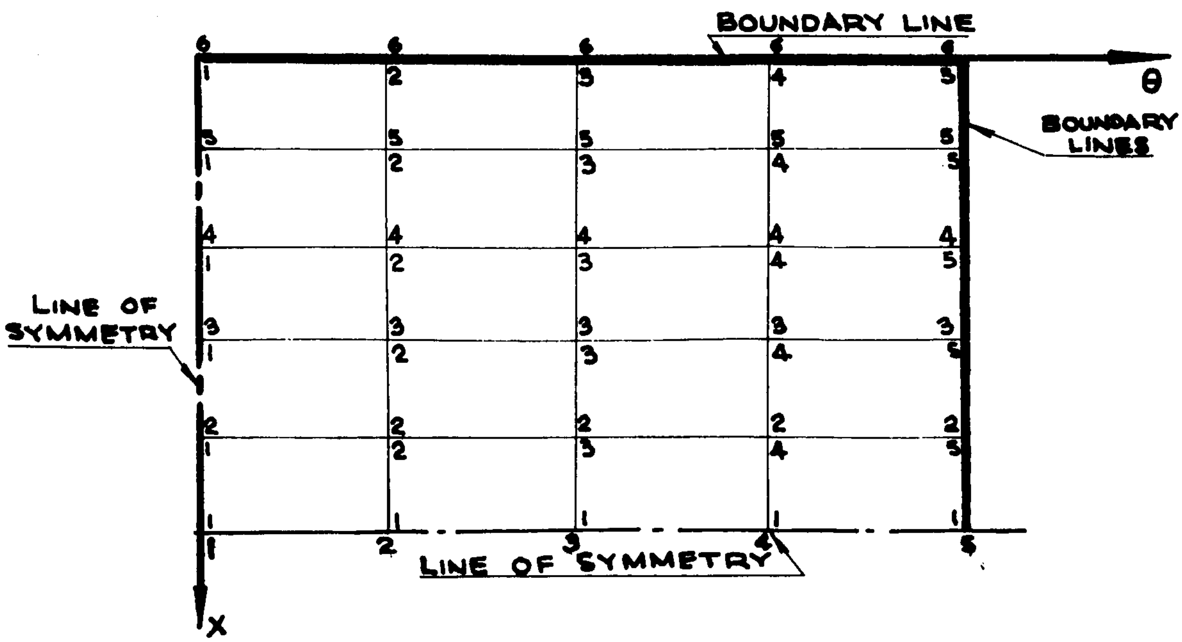

Fig. IV-1 Mesh Point Numbering for Cylindrical Shell Segment even such a small mesh [Fig. IV-1] leads to a system of 60 simultaneous equations while a 30 by 30 mesh generates 2700 equations. A digital computer program to form the finite difference equations corresponding to a rectangular mesh of these dimensions presents no unusual difficulties. The first requirement is to determine the ordering of the equations and the unknowns in the simultaneous equation system. Finite-difference equations [Eq. (6-2-6)] for a general interior point (i , j) are written in terms of the neighboring points $(i + 1 , j)$; $(i - 1 , j)$, $(i , j + 1)$, etc. Thus the coefficients for an equation at a particular point (i , j) are obtained simply by substituting the numerical values of i and j in these general formulas. Special tests must be made to alter the results of this substitution for points near the boundary or symmetry lines.

The general methods available for the solution of the resulting linear system $AX = B$ (e.g. Gauss elimination, etc.) are not practical for such a large number of equations due to excessive storage and time requirements. Iterative methods[1] have been used successfully in solving large systems derived from finite difference approximations. When such techniques are applicable, they frequently provide the most rapid and accurate method of solution. Examination of the equation system $AX = B$ shows that the conditions insuring convergence of the point iterative methods are not satisfied. Several methods have been proposed to modify the matrix equation so that these conditions will be satisfied [see Bodewig[2] and Faddeeva[3]]. However, the resulting increased cost in storage and running time per iteration makes such a procedure impractical in the present situation. More recently, the use of "block" iterative techniques has been investigated by several authors. Unfortunately, the theoretical basis for such techniques has so

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[2] Bodewig, F., Matrix Calculus, North-Holland Publishing Company, Amsterdam, 1959
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far only been provided for specialized problems. Preliminary trials to the present problem indicate that Gauss-Seidel 2-line iteration converges but only very slowly. Thus some method to accelerate convergence along the lines of the various overrelaxation schemes described by Varga[1] must be developed before iteration will become practical. Consequently, while it is felt that iterative methods should be investigated and may ultimately provide the best method, initial efforts have been concentrated on developing a feasible direct solution. A direct method also appears attractive when solutions are desired for the same matrix A with many different right hand vectors B.

IV.2 A direct method for the solution of $AX = B$

The method developed to solve the simultaneous equation system is related to Choleski decomposition. By the Choleski[2] decomposition algorithm, any non-singular matrix can be factored into the product of an upper-triangular and a lower-triangular matrix. The solution vector X can then be readily obtained from the factored form of the matrix. However, storage requirements and the computational effort required preclude the use of this algorithm for large systems. Instead, a factorization of the matrix A into the product of an upper-triangular and a lower-triangular matrix is accomplished by first partitioning A and then devising a recursion algorithm applicable to the submatrices of A rather than to its elements.

The feasibility of this method is directly dependent on the form of the matrix A which in turn is dependent on the ordering of the equations and unknowns. One natural method for obtaining a suitable partitioning is the following one. The three finite difference equations for the first mesh point on the first row $(1, 1)$ become the first three equations of A. The following equations are taken from the remaining points on the first row. The unknowns are ordered in the same manner. Then the equations for the first point on the second row are written, and so on. The matrix A is accordingly partitioned in the manner shown below, where m is the number of rows in the finite difference mesh.

$$A = \begin{bmatrix} E_1 & F_1 & G_1 & 0 & \cdot & \cdot & \cdot & & & 0 \\ D_2 & E_2 & F_2 & G_2 & 0 & \cdot & \cdot & & & \cdot \\ C_3 & D_3 & E_3 & F_3 & G_3 & 0 & \cdot & & & \cdot \\ 0 & C_4 & D_4 & E_4 & F_4 & G_4 & 0 & & & \cdot \\ \cdot & & & & & & & & & 0 \\ \cdot & & & & & & & & & G_{m-2} \\ \cdot & & & & & & & & & F_{m-1} \\ 0 & & & & & & 0 & C_m & D_m & E_m \end{bmatrix}$$

The submatrix F_1, for example, contains coefficients of unknowns from the second row of the mesh which appear in equations for points on the first row of the mesh. The most significant feature of A is that all of the non-zero submatrices of A are contained in five diagonals. This is caused by the fact that for the present problem, the finite difference equations for a given mesh row involve unknowns from mesh rows at most two rows above or below. The matrix A can now be factored into the product of a lower-triangular matrix L and an upper triangular matrix U both of which are partitioned in the same manner as A .

$$A = LU$$

$$L = \begin{bmatrix} J_1 & 0 & \cdot & \cdot & & & & 0 \\ H_2 & J_2 & 0 & \cdot & & & & \cdot \\ C_3 & H_3 & J_3 & 0 & & & & \cdot \\ 0 & C_4 & H_4 & J_4 & 0 & & & \cdot \\ \cdot & & & & & & & 0 \\ 0 & & & & 0 & C_m & H_m & J_m \end{bmatrix} \quad U = \begin{bmatrix} I & M_1 & N_1 & 0 & \cdot & & & 0 \\ 0 & I & M_2 & N_2 & 0 & & & \cdot \\ \cdot & 0 & & & & & & \cdot \\ & \cdot & & & & & & N_{m-2} \\ \cdot & & & & 0 & & I & M_{m-1} \\ 0 & \cdot & & & & & 0 & I \end{bmatrix}$$

The lowest diagonal of L is the same as the corresponding diagonal of A and the main diagonal of U consists of identity submatrices. The remaining submatrices of L and U are obtained from the following recursion scheme:

1. $H_i = D_i - C_i M_{i-2}$ $\qquad i = 2, m$

2. $J_i = E_i - H_i M_{i-1} - C_i N_{i-2}$ $\qquad i = 1, m$

3. $M_i = J_i^{-1} (F_i - H_i N_{i-1})$ $\qquad i = 1, m-1$

4. $N_i = J_i^{-1} G_i$ $\qquad i = 1, m-2$

For each fixed i, the submatrices of L and U are formed in the listed order (when the subscript of a term is less than 1, the corresponding term is zero). The proof that $A = LU$ is obtained directly by multiplication. Of course, it is necessary that the submatrices J_i all be non-singular. When the factorization has been completed, the auxiliary matrix equation $LZ = B$ is solved by the following recursion:

$$Z_i = J_i^{-1} (B_i - H_i Z_{i-1} - C_i Z_{i-2}) \qquad i = 1, m$$

Finally, the equation $UX = Z$ is solved by another recursion:

$$X_{m-i} = Z_{m-i} - M_{m-i} X_{m-1-i} - N_{m-i} X_{m-2-i} \qquad i = 0, m-1$$

(when a subscript is greater than m, that term is zero)

IV.3 A digital computer program for a cylindrical shell segment

A digital computer program has been written for the IBM 7094 using the FORTRAN II, version II language, which solves for the displacements of a cylindrical shell segment with its edges clamped. Assuming a finite difference mesh with an equal number of rows and columns, a system of approximately 3000 equations can be solved. The program is organized into a main program, which controls the sequence of steps in the various recursion schemes, and a collection of subroutines called by the main program in which data is generated or matrix operations are performed, etc. The chief difficulty in programming the method described above is the problem of minimizing memory storage requirements while at the same time keeping the total computation time within reasonable bounds. A 30 by 30 finite difference mesh generates submatrices consisting of 8100 elements. There is therefore room for at most three such submatrices in core storage at a given time. Consequently, a substantial amount of intermediate tape storage is unavoidable. By selecting the sequence of operations carefully and overlapping tape activity with internal computation, the delays caused by tape reading or writing have been reduced to a small fraction of the total time. The program has been designed so that the number of rows and columns may be varied independently. The only limitation on the size of the mesh is the number of columns, which must not exceed 32. Let m be the number of rows and n the number of columns in the finite difference mesh. An approximate formula for the time required on the 7094 is:

$$t = m \cdot n^3/20{,}000 \text{ minutes} .$$

It should be noted that the execution time is directly proportional to the number of rows. If a mesh with many more columns than rows should be required, it would be desirable to rewrite portions of the program to interchange the role of the rows and columns. No essential alteration of the method would be necessary to accomplish this.

The accuracy of the computations has been improved by utilizing the double precision features available on the 7094 for the most critical operations of matrix multiplication and matrix inversion.

Figure IV-2 and Fig. IV-3 present the results of the computer program for displacements along the two lines of symmetry for varying mesh spacing. The loading is due to a uniform pressure normal to the shell surface.

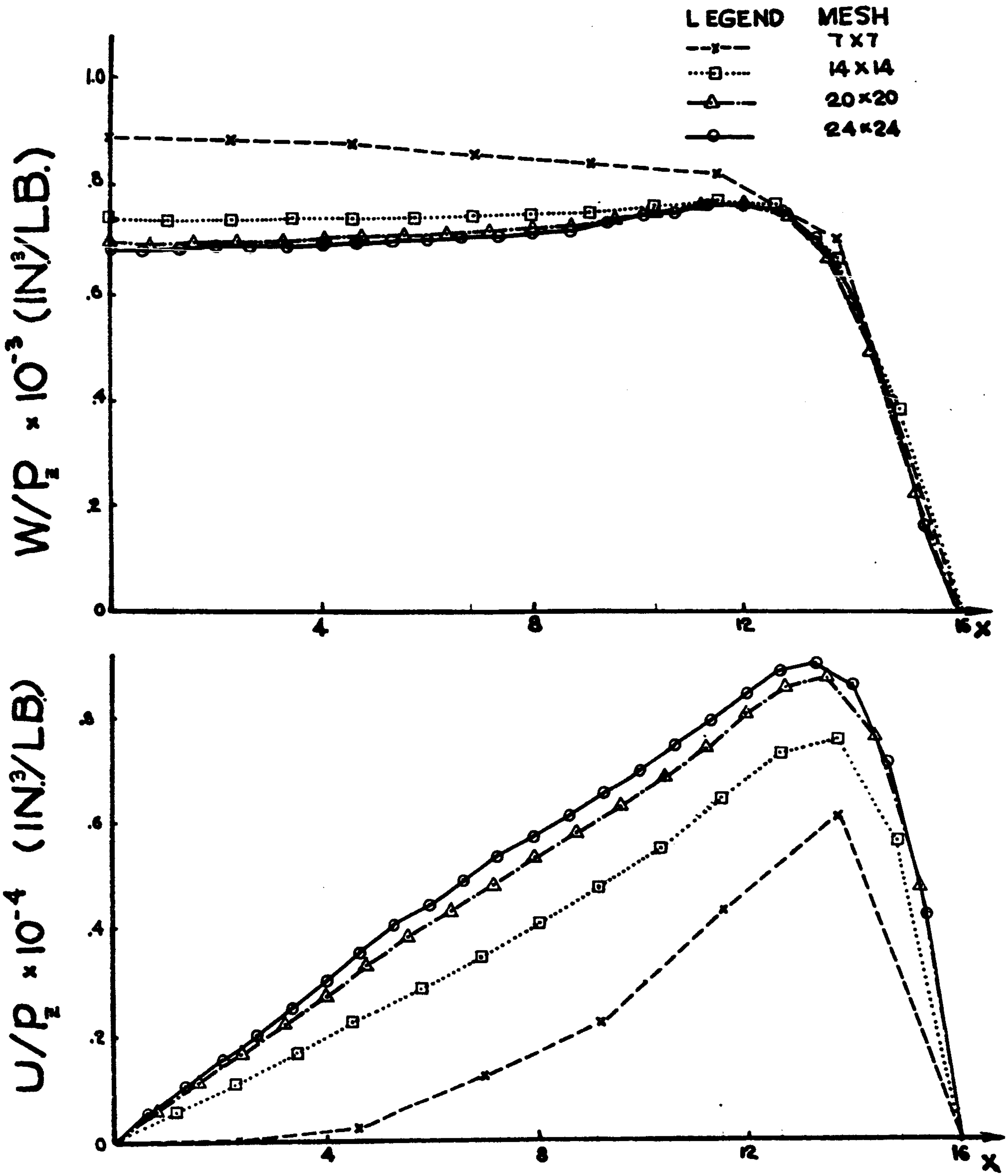

FIG. IV-2
DISPLACEMENT COMPONENTS w AND u ALONG SYMMETRY LINE $\Theta = 0$ FOR VARIOUS MESH SPACING

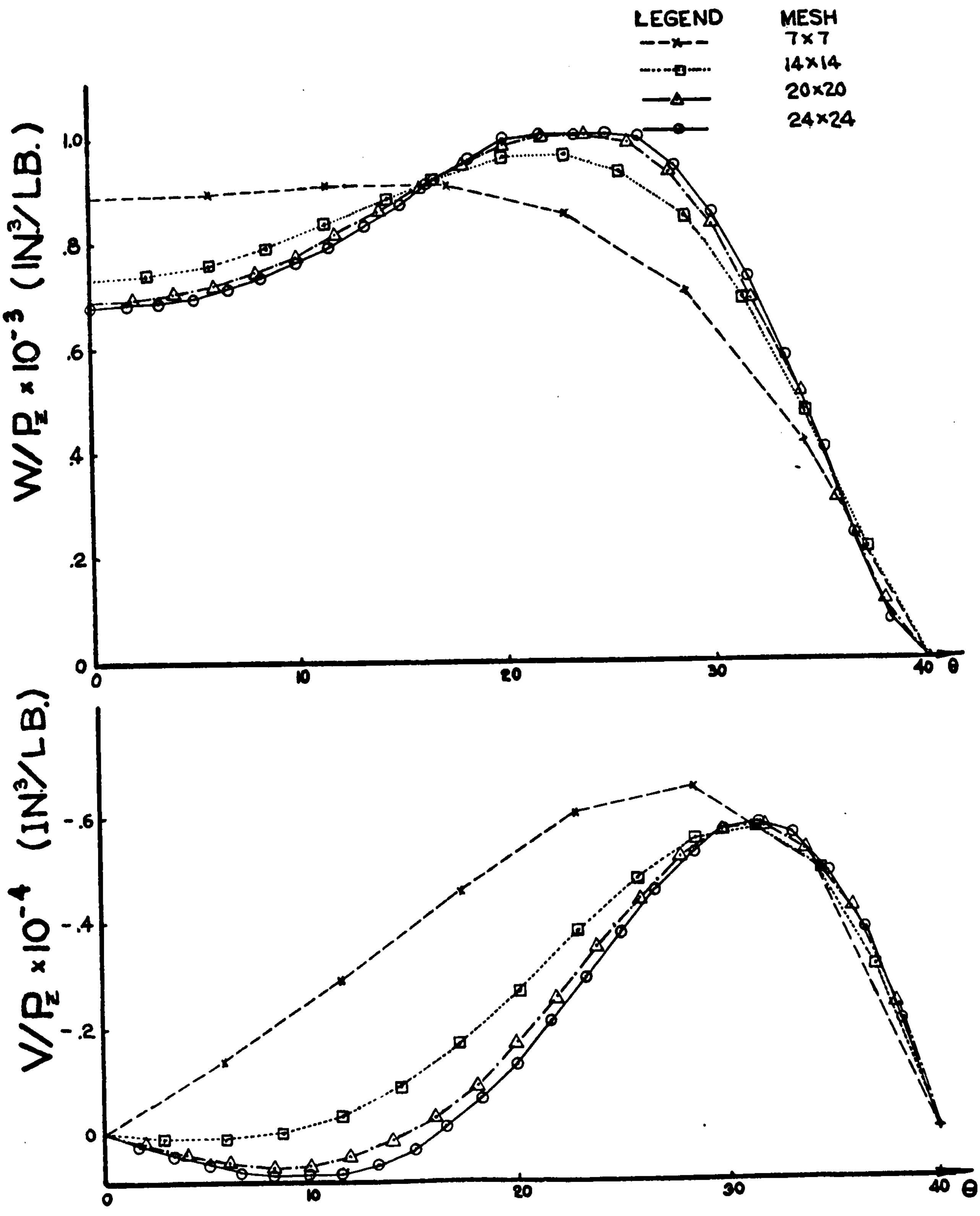

FIG. IV-3

DISPLACEMENT COMPONENTS w AND v ALONG SYMMETRY LINE X=CONSTANT FOR VARIOUS MESH SPACING

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